Conductance Fluctuations in the Ballistic Regime: A Probe of Quantum Chaos?

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We demonstrate the existence of resistance fluctuations in experimentally realizable ballistic conductors due to scattering from geometric features. The magnetic-field and energy correlation functions are calculated both semiclassically and exactly numerically, and are found to have a scale determined by the underlying chaotic classical scattering. These systems provide a test of the "random" quantum behavior of classically chaotic systems.

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A signature of chaos in a classical dynamical system is "random" (unpredictable) behavior due to an exponential sensitivity to initial conditions. It is now understood that quantum systems whose classical analogs are chaotic also exhibit "randomness" in a more subtle manner. For example, the energy levels of a Hamiltonian begin to exhibit the long-range level repulsion[1] characteristic of random matrices[2] at the classical transition to chaos. Recent studies of "quantum chaos" have turned attention to open systems whose classical scattering dynamics is irregular or chaotic in the sense that, e.g., the final scattering angle varies strongly with the incident angle on an *arbitrarily* fine scale. In these systems, the signature of quantum chaos is fluctuations of the quantum scattering matrix with properties described by random-matrix theory.[3] Since the transport properties of small, phase-coherent conductors may be expressed completely in terms of the elements of the S matrix for independent electrons at the Fermi energy E_F,[4] this raises the possibility of an experimental probe of this random behavior characteristic of quantum chaos. In this Letter, we show that *ballistic* conductors of the type which have been extensively studied recently[4] do indeed show fluctuations [very similar to the universal conductance fluctuations (UCF) of bulk-disordered metals[5]], whose statistical properties can be predicted from properties of the chaotic classical scattering dynamics.

It is now possible to fabricate two-dimensional electronic conductors with controllable geometric features much smaller than the elastic or (at $T \leq 4$ K) inelastic scattering length (l_{in}). Hence electron motion is ballistic and resistance arises primarily from scattering from geometric features such as the junction of four wires. In such junctions novel magnetotransport effects[4] appear at relatively low magnetic field ($B \leq 2$ T) and $T \sim 4$ K, which have been shown to depend on the junction geometry.[4,6] Recently, Beenakker and van Houten[7] showed that most of the generic effects observed could be reproduced simply from the *classical* S matrix of the junction. However, there was a very significant difference between the quantum transport coefficients calculated in Ref. 6 and the classical ones of Ref. 7, shown in Fig. 1. The $T=0$ quantum coefficients show aperiodic structure as a function of B on a scale much smaller than that of the classical features such as the suppression ("quenching"[4,8]) of the Hall resistance. Reference 6 found simiiar fluctuations as a function of E_F. Such *weak-field* fluctuations are in fact observed in many relevant experiments at $T \approx 0.1$ K,[9] although it was not possible to rule out impurity effects as their origin. Our calculations show that aperiodic structure (similar to UCF) can

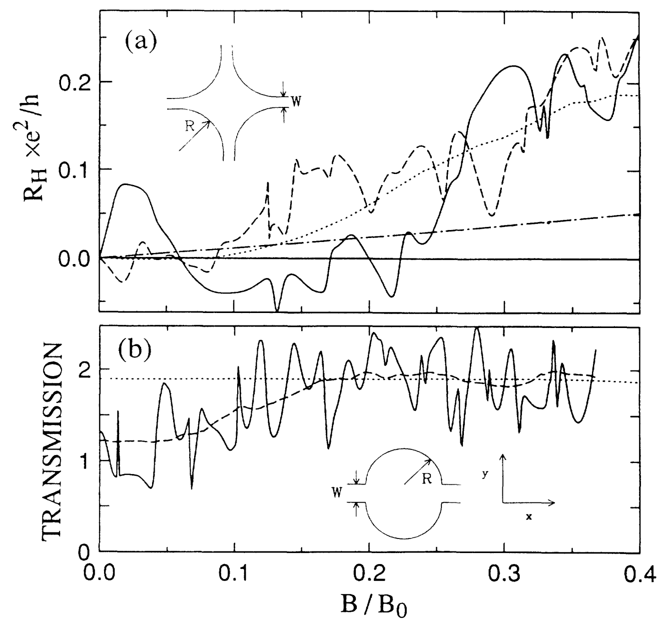

FIG. 1. (a) Hall resistance for four-disk junction with $R/W=4$ (inset). The two quantum calculations with slightly different E_F [solid (dashed) line for $k_F W/\pi = 4.3$ (4.4)] show fluctuations not present in either the classical (dotted) or quantum square junction ($R/W=0$, dash-dotted) cases. (b) $T(B)$ for open stadium with $R/W=2$ (inset). The solid line is the quantum result ($k_F W/\pi = 4.5$), the dashed line is smoothed $\langle T(B) \rangle$ used in computing $C(\Delta B)$, and the dotted line is the classical result. $B_0 = mcv_F/eW$.

occur in systems such as a junction or cavity with *no bulk disorder*. Hence the quantum scattering from such simple objects generates as much complexity in the transport coefficients as would a random potential; this is an essential notion in chaos theory, and below we establish the connection explicitly. The fluctuations arise due to the complex scattering dynamics from such geometric features which can trap the electrons for times much longer than the ballistic transit time. Long trapping times are essential because, e.g., the Hall resistance of a square junction, which cannot trap particles, shows no such behavior [Fig. 1(a)].

We will now show that the characteristic scale in B or E_F for these fluctuations in the *quantum* transport coefficients can be predicted from knowledge of the irregular *classical* scattering dynamics of the same system. Our results are in strong contrast to recent work which has discussed fluctuations in the *classical* transmission properties.[7,10] The results in Fig. 1 show that fluctuation effects in the classical transport coefficients (calculated as in Ref. 7) are completely absent in these structures, at these fields. The fluctuations of Fig. 1 are a quantum *interference* effect, not obtainable from the classical S matrix.

We consider the scattering from two types of open "billiards," a two-probe system [Fig. 1(b)] consisting of two separated semicircles (an open "stadium") and a four-probe system [Fig. 1(a)] consisting of four quarter disks with leads. In both cases, the trapping time τ of injected particles increases with R/W, the ratio of the radius of the circle to the width of the leads. The classical scattering from such billiards is chaotic;[3,7,10] for example, a plot of τ versus incident angle shows irregular regions in which a complex pattern reproduces itself self-similarly on an arbitrarily fine angular scale. For such systems it is known that the number of injected particles remaining in the scattering region after time τ satisfies

$$N(\tau)=N(0)\exp(-\tilde{\gamma}_{cl}\tau)\,, \tag{1}$$

where $\tilde{\gamma}_{cl}=\lambda(1-d)$ is the classical escape rate, λ is the Liapunov exponent of the manifold of infinitely trapped orbits (strange repeller), and d is the information dimension of the unstable manifolds.[3,11] Although invalid for the shortest trajectories, the exponential law holds as $\tau\to\infty$, due to the presence of scattering paths arbitrarily close to infinitely trapped unstable orbits. $\tilde{\gamma}_{cl}$ is found to be independent of the distribution of incident particles (barring degenerate choices), and for billiards (in which the speed v of the scattering particle is fixed) we may for convenience define an inverse escape length $\gamma_{cl}\equiv\tilde{\gamma}_{cl}/v$ and study the distribution of trajectory lengths, $N(L)$. We have numerically determined $N(L)$ and hence γ_{cl} for these billiards [inset, Fig. 3(a)]. Typical paths are not exponentially long, but correspond to $\sim$5–15 transits; thus in the quantum problem we are not in the regime of transmission resonances. Nonetheless, from the time-energy uncertainty relation, it is natural to suppose that the energy scale for the fluctuations is of order $\hbar\tilde{\gamma}_{cl}=\hbar\langle 1/\tau\rangle$. Such a relationship has, in fact, been demonstrated recently by Blümel and Smilansky[3] on the basis of a semiclassical derivation of the energy correlation function. Expressing their result for the two-probe case [Fig. 1(b)] in terms of wave vector $k=(2mE/\hbar^2)^{1/2}$ gives

$$C(\Delta k)=\frac{C(0)}{1+(\Delta k/\gamma_{cl})^2}\,, \tag{2}$$

where $C(\Delta k)=\langle\delta g(k+\Delta k)\delta g(k)\rangle$ averaged over an appropriate k interval,[3] and the conductance $g=(e^2/h)T$, with T the total transmission coefficient. $T=\sum_{n,m=1}^{N}|t_{mn}|^2$, where t_{mn} is the transmission amplitude between the N transverse modes in the leads.

For applications to ballistic conductors the more relevant quantity is the magnetic-field correlation function[5] $C(\Delta B)$, since B is the most convenient experimental control parameter. We outline a semiclassical derivation of $C(\Delta B)$ below, focusing on the two-probe case for simplicity; details will be given elsewhere.[12] An exact starting point is[4]

$$t_{nm}=-i\hbar(v_m v_n)^{1/2}\int dy'\int dy\,\psi_n^*(y')\psi_m(y)G(y',y,E_F)\,, \tag{3}$$

where v_m (v_n) and ψ_m (ψ_n) are the longitudinal velocity and transverse wave function for the mode m (n). G is the retarded Green's function, between points (x,y) on the left lead and (x',y') on the right lead (omitting an irrelevant phase factor dependent on x,x'). To approximate t_{mn} we replace G by its semiclassical path-integral expression,[13]

$$G(y',y,E)=\frac{2\pi}{(2\pi i\hbar)^{3/2}}\sum_{s(y,y')}\sqrt{D_s}\exp\left[\frac{i}{\hbar}S_s(y',y,E)-i\frac{\pi}{2}\mu_s\right], \tag{4}$$

where S_s is the action integral along *classical* path s, at energy E, $D_s=(v\cos\theta'/m)^{-1}|(\partial\theta/\partial y')_y|$, θ and θ' are the incoming and outgoing angles, and μ is the Maslov[13] index given by the number of constant-energy conjugate points. The restriction to paths at energy E arises from a stationary-phase approximation in the time Fourier transform of the WKB propagator.[13] We evaluate t_{mn} from Eqs. (3) and (4) in an analogous manner. Assuming hard walls in the leads, the integration over the transverse (sine) wave functions can be performed by use of the stationary-phase approximation, valid in the (semiclassical) $\hbar\to 0$ limit. For example, the stationary-phase condition on the y integration is $(\partial S/\partial y)_{y'}=-p_y=-\bar{m}\hbar\pi/W$ for $y=y_0$ ($\bar{m}=-m,m$); i.e., the classical trajectories that contribute are those for which the incident transverse momentum is equal to the appropriate quantized value. We obtain

$$t_{nm}=-\frac{(2\pi i\hbar)^{1/2}}{2W}\sum_{s(\bar{m},\bar{n})}\mathrm{sgn}(\bar{m})\mathrm{sgn}(\bar{n})\sqrt{\tilde{D}_s}\times\exp\left[\frac{i}{\hbar}\tilde{S}_s(\bar{n},\bar{m},E)-i\frac{\pi}{2}v_s\right]. \tag{5}$$

The sum is now over trajectories between the cross sections x and x' at angles $\sin\theta=\bar{m}\pi/kW$ and $\sin\theta'=\bar{n}\pi/kW$. The action is

$$\tilde{S}(\bar{n},\bar{m},E)=S(y_0',y_0,E)+\hbar\pi\bar{m}y_0/W-\hbar\pi\bar{n}y_0'/W,$$

the prefactor is

$$\tilde{D}_s=(mv\cos\theta')^{-1}|(\partial y/\partial\theta')_\theta|,$$

and

$$v=\mu+H(-(\partial\theta/\partial y)_{y'})+H(-(\partial\theta'/\partial y')_\theta)$$

(H is the Heaviside step function).[14]

$C(\Delta B)=\langle\delta g(B+\Delta B)\delta g(B)\rangle$ can be calculated using Eq. (5) by arguments similar to Ref. 3. First, in the multiple sum over paths, only the interference of a path s at field B with the same path at $B+\Delta B$ is retained. Second, the difference of the action is expanded to find the relative phase accumulated:

$$\frac{1}{\hbar}\left(\frac{\partial\tilde{S}_s}{\partial B}\right)\Delta B=\frac{e}{\hbar c}\int\Delta\mathbf{A}\cdot dl\equiv\frac{\Theta_s\Delta B}{\phi_0}, \tag{6}$$

where $|\nabla\times\Delta\mathbf{A}|=\Delta B$, $\phi_0=hc/e$. Unlike τ, Θ can be positive or negative (see Fig. 3), and we find that for weak fields, except near the origin, Θ has an approximately symmetric exponential distribution, $N(\Theta)\propto\exp(-\alpha_{\rm cl}\times|\Theta|)$. *We emphasize that the entire analysis is only valid for weak fields, where the cyclotron radius is larger than the device dimensions.* For closed orbits, Θ is the area enclosed (times 2π). For typical open orbits this is true to a good approximation, and the inverse decay constant $\alpha_{\rm cl}^{-1}$ will then give the root-mean-squared area enclosed by trajectories traversing the structure (although Θ is not itself gauge invariant, we have checked that $\alpha_{\rm cl}$ is independent of gauge[12]). Finally, making the approximation $N(\Theta)\propto\exp(-\alpha_{\rm cl}|\Theta|)$ for *all* Θ we can replace the sum over trajectories by an integral over Θ, giving

$$C(\Delta B)=C(0)/[1+(\Delta B/\alpha_{\rm cl}\phi_0)^2]^2. \tag{7}$$

We have calculated by classical simulations $\gamma_{\rm cl},\alpha_{\rm cl}$ for the structures shown, and hence can compare the predictions of Eqs. (2) and (7) to exact numerical results for the correlation functions obtained by the recursive Green's-function method[6] with no free parameters. We consider systems with 4–15 modes at E_F. It is important to note that the semiclassical approximation (SCA) leading to Eq. (4) or (5) is *not* well justified in the few-mode limit typical of experiments, since the difference between the actions of classical trajectories with the same end points is not always much greater than $\hbar$. Thus our numerical results provide a crucial test of the accuracy of the SCA in such systems.

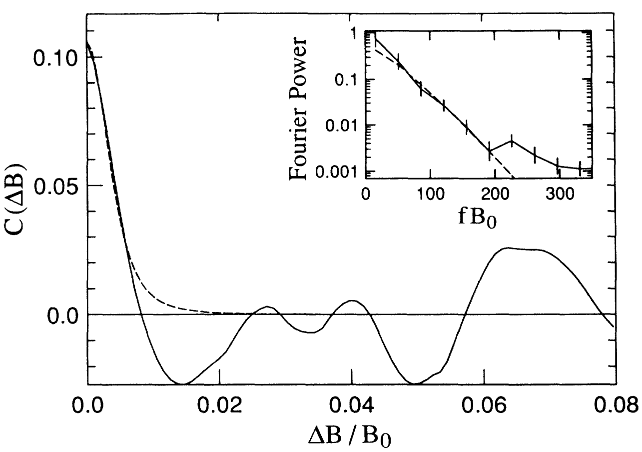

FIG. 2. Magnetic-field correlation function from the data of Fig. 1(b) (for $W=1000$ Å, half-width $\Delta B_c=40$ G). The dashed line is the semiclassical prediction of Eq. (7). Inset: The smoothed power spectrum of $T(B)$. Error bars indicate the rms variation of raw data. The dashed line is the best fit to the Fourier transform of Eq. (7) in the interval [50,200].

First we show (Fig. 2) a typical correlation function $C(\Delta B)$ and the SCA prediction of Eq. (7). The agreement is excellent except in the tail which corresponds to the nonuniversal short-trajectory behavior. Because the behavior in the tail affects the half-width, it is necessary to calculate δT from a smoothed $\langle T(B)\rangle$ curve [Fig. 1(b)], which introduces some arbitrariness in the determination of α_{qm}. To eliminate this freedom, we instead extract α_{qm} from fitting the Fourier power spectrum of the data by the Fourier transform of Eq. (7) (inset of Fig. 2); the value thus extracted is independent of the nonuniversal "low-frequency" behavior. The same approach was used to extract γ_{qm} from $C(\Delta k)$, by fitting with the simple linear power spectrum of Eq. (2).

Figure 3 shows that γ_{qm},α_{qm} are indeed given by the classical quantities $\gamma_{\rm cl},\alpha_{\rm cl}$ to high accuracy while they are varied over roughly 2 orders of magnitude by changing R/W in the two- and four-probe structures. *Thus it is possible to predict quantitatively measurable properties of these ballistic quantum conductors from a knowledge of the chaotic classical scattering dynamics.*

A simple heuristic argument based on the SCA above implies that the *magnitude* of the transmission fluctuations, $C(0)$, is order unity (see, e.g., Fig. 2), independent of N and R/W, and detailed numerical studies confirm this.[12] Thus the effect at $T=0$ is large; however, we expect the magnitude to be *exponentially* sensitive to temperature because, unlike the diffusive case, since once $l_{\rm in}$ is of the order of the system size, it is impossible to break

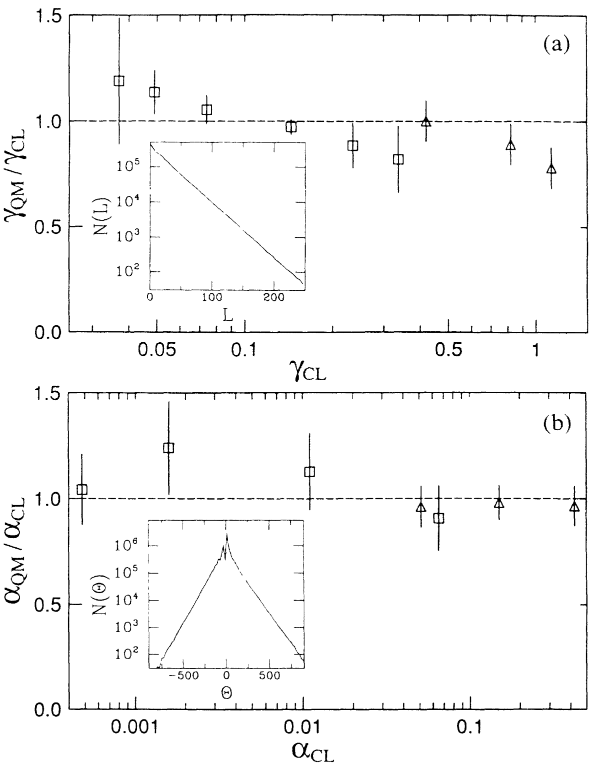

FIG. 3. (a) Ratio of the wave-vector correlation length, obtained by fitting power spectrum as discussed in text, to classical escape rate γ_{cl} as a function of γ_{cl} for both structures. Four-disk structure (triangles) with $R/W=1,2,5$, and open stadium (squares) with $R/W=0.5,1,2,4,6,8$. γ_{cl} is obtained from the classical $N(L)$ curve and is in units of the distance between opposite leads; L is measured in units of W. (b) Ratio of magnetic-field correlation length (obtained as in inset of Fig. 2) to α_{cl}, the exponent of the distribution of effective areas, as a function of α_{cl} for both structures. Four-disk structure (triangles) with $R/W=1,2,4$, and open stadium (squares) with $R/W=1,2,4,6$. α_{cl} (in units of W^{-2}) is obtained from classical $N(\Theta)$ curve. Insets: (a) $N(L)$ and (b) $N(\Theta)$ for structure of Fig. 1(b). The correlation lengths of the quantum fluctuations agree with the semiclassical prediction over two decades.

up the system into smaller phase-coherent units which show the effect. Moreover, the important trajectories traverse the system several times; thus a reasonable criterion for the crossover temperature is $\gamma_{cl} l_{in}(T) \approx 1$.

We briefly comment on two important issues to be treated elsewhere.[12] First, the nonuniversal behavior due to short trajectories does cause very-low-frequency peaks in the Fourier spectra whose intensities depend on the mode index (γ_{qm},α_{qm} do not), and increase with decreasing total number of modes. For example, T_{11} has a strong periodic component in the open stadium with five modes; this is due to the caustic associated with direct paths.[12,14] Second, there is the fundamental question of the extent to which these fluctuation effects are a *unique* signature of quantum chaos. We have performed exactly the same analysis of scattering from nonchaotic cavities, e.g., a rectangular box, and still find substantial aperiodic structure. However, a careful analysis of the Fourier spectra indicates that they have a qualitatively different and nonuniversal shape in the many-mode limit, which is clearly distinguishable given 2- or 3-orders-of-magnitude sensitivity.

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Note added.—After completion of this work we received a preprint by E. Doron, U. Smilansky, and A. Frenkel, applying an analysis similar to that of Fig. 3(a) to measurements of the frequency-dependent reflectivity from a microwave cavity.

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