The Multi-Agent Rendezvous Problem - Part 2
The Asynchronous Case*

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Abstract

This paper is concerned with the collective behavior of a group of $n > 1$ mobile autonomous agents, labelled 1 through n, which can all move in the plane. Each agent is able to continuously track the positions of all other agents currently within its "sensing region" where by an agent's *sensing region* is meant a closed disk of positive radius r centered at the agent's current position. The *multi-agent rendezvous problem* is to devise "local" control strategies, one for each agent, which without any active communication between agents, cause all members of the group to eventually rendezvous at single unspecified location. This paper describes a family of unsynchronized strategies for solving the problem. Correctness is established appealing to the concept of "analytic synchronization."

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1 Introduction

This paper is concerned with the collective behavior of a group of $n > 1$ mobile autonomous agents, labelled 1 through n, which can all move in the plane. Each agent is able to continuously track the positions of all other agents currently within its "sensing region" where by an agent's *sensing region* is meant a closed disk of positive radius r centered at the agent's current position. The *multi-agent rendezvous problem* is to devise "local" control strategies, one for each agent, which without any active communication between agents, cause all members of the group to eventually rendezvous at single unspecified location.

The rendezvous problem, which is also sometimes called a "gathering problem," has been studied before assuming that all agents possess either unlimited visibility {e.g., $r = \infty$} [2], or a common coordinate system [3] or both. The problem has also been addressed before without making either of these assumptions [4, 5]. This paper also treats the case which individual agents have limited visibility and distinct frames of reference. What distinguishes this work from that in [4, 5] is that individual agents clocks are taken to be unsynchronized. These three features, namely limited sensing, no common frame of reference, and no common clock, are of obvious practical importance but have apparently not been dealt with before all at once as components of one multi-agent rendezvous problem.

As in [4, 5], we consider distributed strategies which guide each agent toward rendezvous by performing a sequence of "stop-and-go" maneuvers. A *stop-and-go maneuver* takes place within a time interval consisting of two consecutive sub-intervals. The first, called a *sensing period*, is an interval of fixed length during which the agent is stationary. The second, called a *maneuvering period*, is an interval of variable length during which the agent moves from its current position to its next 'way-point' and again come to rest. Successive way-points for each agent are chosen to be within r_M units of each other where r_M is a pre-specified positive distance no larger than r. It is assumed that there has been chosen for each agent i, a positive number τ_{M_i}, called a *maneuver time*, which is large enough so that the required maneuver for agent i from any one way-point to the next can be accomplished in at most τ_{M_i} seconds. Since our interest here is exclusively with devising of *high level* strategies which dictate when and where agents are to move, we will use point models for agents and shall not deal with how maneuvers are actually carried out or with how vehicle collisions are to be avoided.

In the synchronous case treated in [4, 5], the *kth* maneuvering period of each agent is synchronized to begin at the same time $\bar{t}_k$ as the *kth* maneuvering period of every other agent. Agent i's registered neighbors at the beginning of its *kth* maneuvering period are taken to be all those other agents positioned within agent i's sensing region at the beginning of the period. Because of synchronization, this notion of a registered neighbor induces a *symmetric* relation on the agent group in that agent j is a registered neighbor of agent i at the beginning of maneuvering period k just in case agent i is a registered neighbor of agent j at the same time. As a result, it is possible to characterize neighbor relationships at time $\bar{t}_k$ with a simple graph whose vertices represent agents and whose edges represent existing neighbor relationships [5]. Although the neighbor relation is symmetric, it is clearly not transitive. On the other hand if agent i is at the same position as neighbor j at time $\bar{t}_k$, then any registered neighbor of agent j at time $\bar{t}_k$ certainly must be a registered neighbor of agent i at the same time. It is precisely because of this *weak transitivity* property that one can infer a *global* condition of the entire synchronized agent group from a *local* condition of one agent and its neighbors. In particular, if the graph characterizing neighbor relationships at time $\bar{t}_k$ is connected, and any one agent is at the same position as all of its neighbors, then the weak transitivity property guarantees at once that all n agents have rendezvoused at time $\bar{t}_k$.

Our aim in this paper is to relax the synchronization requirement. In particular we will not require synchronization of the start times of the maneuvering periods of different agents. To accomplish this it is necessary to modify somewhat what is meant by a registered neighbor of agent i at time $\bar{t}_{ik}$ where for the asynchronous case under consideration, $\bar{t}_{ik}$ denotes the time at which agent i's kth maneuvering period begins. Our definition is guided by considerations discussed above for the synchronous case. For example, the new definition is crafted to retain versions of the symmetry and weak transitivity properties of the registered neighbor relation inherent in the synchronous case. Doing this is challenging, because unlike the synchronous case, the times each agent registers its neighbors and its neighbors' positions are not synchronized with the times its neighbors do the same thing.

Exactly the same way-point update rules considered in the synchronous case [5] are adopted for the asynchronous case. Thus the only functional differences between the two cases are the definitions of registered neighbors and registered neighbor positions. Of course in the asynchronous case, way-point updates are computed asynchronously, whereas in the synchronous case they are not.

Not surprisingly, the analysis of the asynchronous version of the problem is considerably more challenging than that of the synchronous version. For example, while it is more or less obvious in the synchronous case that the proposed way point update rules cause all agents retain their neighbors as the system evolves [5], proving that this is also true in the asynchronous case involves a number of steps.

Just as in the synchronous case, it is possible to characterize neighbor relationships with a graph. This is done in §3 by first merging together into a single ordered time set the distinct "event times" $\bar{t}_{ik}$, $i \in \{1, 2, \ldots, n\}$, $k \geq 1$ generated by all n agents. The elements of this set are then relabelled as $t_1, t_2, \cdots$ in such a way so that $t_j < t_{j+1}$, $j \in \{1, 2, \ldots\}$. With this notation, agent i's registered neighbors at its kth event time $\bar{t}_{ik}$ are its registered neighbors at time $t_{P_i(k)}$ where $P_i(k)$ denotes that value of p for which $t_p = \bar{t}_{ik}$. For each $i \in \{1, 2, \ldots, n\}$, the domain of definition of agent i's registered neighbors is then extended from the set $\{t_{P_i(k)} : k \geq 1\}$ to the set $\{t_p : p \geq P_i(1)\}$ by stipulating that for values of t_p which are between two successive event times of agent i, say between $\bar{t}_{ik}$ and $\bar{t}_{i(k+1)}$, agent i's registered neighbors are the same as its registered neighbors at time $\bar{t}_{ik}$. This means that registered neighbors of each agent are defined at each time $t_p \geq t_{\bar{p}}$ where $\bar{p} \stackrel{\Delta}{=} \max\{P_1(1), P_2(1), \ldots P_n(1)\}$. Because of this, it is possible to describe neighbor relationships with a directed graph with vertex set $\{1, 2, \ldots, n\}$ and directed edge set defined so that (i, j) is a directed edge from vertex i to vertex j just in case agent j is a registered neighbor of agent i at event time t_s. The main result of this paper {Corollary 1} is that if this graph is ever strongly connected, then rendezvous of all n agents will eventually occur.

To establish the correctness of Corollary 1 requires the analysis of the asymptotic behavior of the *asynchronous* process which describe the n-agent system. Despite the apparent complexity of this process, it is possible to capture its salient features using a suitably defined *synchronous* discrete-time, hybrid dynamical system $\mathbb{S}$. We call the sequence of steps involved in defining $\mathbb{S}$ *analytic synchronization*. Analytic synchronization is applicable to any finite family of continuous or discrete time dynamical processes $\{\mathbb{P}_1, \mathbb{P}_2, \ldots, \ldots, \mathbb{P}_n\}$ under the following conditions. First, each process $\mathbb{P}_i$ must be a dynamical system whose inputs consist of functions of the states of the other processes as well as signals which are exogenous to the entire family. Second, each process $\mathbb{P}_i$ must have associated with it an ordered sequence of event times $\{t_{i1}, t_{i2}, \ldots\}$ defined in such a way so that the state of $\mathbb{P}_i$ at event time $t_{i(k_i+1)}$ is uniquely determined by values of the exogenous signals and states of the

$\mathbb{P}_j, \;\; j \in \{1, 2, \ldots, n\}$ at event times t_{jk_j} which occur prior to $t_{i(k_i+1)}$ but in the finite past. Event time sequences for different processes need not be synchronized. Analytic synchronization is a straight forward procedure for creating a single synchronous process for purposes of analysis which captures the salient features of the original n asynchronously functioning processes. As a first step, all n event time sequences are merged into a single ordered sequence of even times $\mathcal{T}$. The "synchronized" state of $\mathbb{P}_i$ is then defined to be the original of $\mathbb{P}_i$ at $\mathbb{P}_i$'s event times $\{t_{i1}, t_{i2}, \ldots\}$; at values of $t \in \mathcal{T}$ between event times t_{ik_i} and $t_{i(k_i+1)}$, the synchronized state of $\mathbb{P}_i$ is taken to be the same at the value of its original state at time t_{ik}. What results is a synchronous dynamical system evolving on $\mathcal{T}$ with state composed of the synchronized states of the n individual processes under consideration. The definition of $\mathbb{S}$ in section §4.1 illustrates the analytic synchronization procedure.

2 The Asynchronous Agent System

The strategy analyzed in [4, 5] cannot be regarded as truly distributed because each agent's decisions must be synchronized to a common clock shared by all other agents in the group. In the sequel we redefine the strategies so that a common clock is not required. To do this it will be necessary to modify somewhat what is meant by a registered neighbor and by a registered neighbor's position.

For each agent i, the real time axis can be partitioned into a sequence of time intervals $[0, t_{i1})$, $[t_{i1}, t_{i2})$, ..., $[t_{i(k-1)}, t_{ik})$, ... each of length at most $\tau_D + \tau_{M_i}$ where τ_D is a number greater than τ_{M_i} called a *dwell time*. Each interval $[t_{i(k-1)}, t_{ik})$ consists of a *sensing period* $[t_{i(k-1)}, \bar{t}_{ik})$ of fixed length τ_D during which agent i is stationary, followed by a maneuvering period $[\bar{t}_{ik}, t_{ik})$ of length at most τ_{M_i} during which agent i moves from its current position to its next way-point. Although all agents use the same dwell time, they operate asynchronously in the sense that the time sequences $t_{i1}, t_{i2}, \cdots, \;\; i \in \{1, 2, \ldots, n\}$ are uncorrelated. Thus each agent's strategy can be implemented independent of the rest, without the need for a common clock.

2.1 Registered Neighbors

Because of the asynchronous nature of the control strategies under consideration, care must be exercised in defining what is meant by a registered neighbor if one is to end up with something similar to the symmetry property of the neighbor relationship defined in the synchronous case. For the asynchronous case, agent i's *registered neighbors* at time $\bar{t}_{ik}$ {i.e., at the beginning of its kth maneuvering period $[\bar{t}_{ik}, t_{ik})$} are taken to be those agents which are fixed at one position within agent i's sensing region for at least $\tau_S > 0$ seconds during agent i's kth sensing period $\mathcal{S}_i(k) \stackrel{\Delta}{=} [t_{i(k-1)}, \bar{t}_{ik})$. Here τ_S is a positive number called a *sensing time*. For reasons to be made clear below, we shall require τ_S to satisfy

$$\tau_S \leq \frac{1}{2}(\tau_D - \tau_{M_i}) \qquad \forall i \in \{1, 2, \ldots, n\} \tag{1}$$

For any agent j, there may be more than one distinct interval of length at least τ_S within $\mathcal{S}_i(k)$ during which agent j is stationary. Let t^* denote the end time of the last of these. For purposes of calculation, agent i takes the *registered position* of agent j at the beginning of its kth maneuvering period, to be the actual position of agent j at *registration time* t^*. To attain a symmetry-like property for the asynchronous case, it is necessary make sure that the *registration interval* $[t^* - \tau_S, \; t^*)$ lies within one of agent j's sensing periods. One way to guarantee that this is so is to require each agent to keep moving during each of its maneuvering periods except possibly for brief periods which are each shorter than τ_S. Another way is equip each agent with a signaling device {such as a light in the case of visual sensing} which is on just in case the agent is in one of its sensing periods. In the sequel we will assume that registration of each agent j during one of agent i's sensing periods always occurs at the end of a registration interval $[t^* - \tau_S,\ t^*)$ which also lies within one of agent j's sensing periods. Note that this and the requirement that agent j is stationary during its sensing periods together imply that agent j's registered position $x_j(t^*)$ is equal to $x_j(\bar{t}_{jk^*})$ where k^* is the sensing/maneuvering interval of agent j during which registration takes place.

2.1.1 Neighbor Characterization

Prompted by the preceding, let us agree to say that for each $i, j \in \{1, 2, \dots, n\}$, agent j's *pth* sensing period $\mathcal{S}_j(p)$ *strongly overlaps* agent i's *kth* sensing period $\mathcal{S}_i(k)$ if the overlap $\mathcal{S}_j(p) \cap \mathcal{S}_i(k)$ is a non-empty interval of length at least τ_S seconds. In the sequel we write $\mathcal{S}_j(p) \cap \mathcal{S}_i(k) \succ \tau_S$ whenever $\mathcal{S}_i(k)$ and $\mathcal{S}_j(p)$ strongly overlap. Let us note that because all sensing periods of all agents are τ_D seconds long, the largest number of sensing periods of any one agent which a given sensing period of agent i can overlap, is two. On the other hand, each sensing period of agent i must strongly overlap at least one sensing period of each other agent. To understand why this is so, note first that the maximal possible amount of time between two successive sensing periods of agent j is τ_{M_j}; but τ_{M_j} is bounded above by $\tau_D - 2\tau_S$ because of (1). Thus the maximal possible amount of time between two successive sensing periods of agent j is no greater than $\tau_D - 2\tau_S$. Given this and the fact that all sensing periods are τ_D seconds long, it follows that each sensing period of agent i must strongly overlap at least one sensing period of each agent j.

It is possible to be more explicit about which sensing periods of agent j overlap $\mathcal{S}_i(k)$. For each $i, j \in \{1, 2, \dots, n\}$ and each $k \geq 1$, let $\lceil \bar{t}_{ik} \rceil_j$ denote the smallest integer q such that $\bar{t}_{jq} \geq \bar{t}_{ik}$. In other words, $\lceil \bar{t}_{ik} \rceil_j$ is the unique integer for which $\bar{t}_{ik} \in (\bar{t}_{j(q-1)}, \bar{t}_{jq}]$. Set $q = \lceil \bar{t}_{ik} \rceil_j$. In view of the definition of $\lceil \cdot \rceil_j$ and the preceding discussion it is clear that the only sensing periods of agent j which $\mathcal{S}_i(k)$ can overlap are $\mathcal{S}_j(q-1)$ and $\mathcal{S}_j(q)$; moreover $\mathcal{S}_i(k)$ must strongly overlap one of these. There are three possible situations which might occur. In the first, shown in Figure 1a, the only sensing period of agent j which overlaps $\mathcal{S}_i(k)$ is $\mathcal{S}_j(q-1)$; in this case the length of the overlap is $\tau_D - (\bar{t}_{ik} - \bar{t}_{j(q-1)})$ and this length will always be greater than or equal to τ_S. Therefore in this situation, $\mathcal{S}_i(k)$ and $\mathcal{S}_j(q-1)$ strongly overlap. For the second situation, shown in Figure 1b, the only sensing period of agent j which overlaps $\mathcal{S}_i(k)$ is $\mathcal{S}_j(q)$; in this case the length of the overlap is $\tau_D - (\bar{t}_{jq} - \bar{t}_{ik})$ and this length will also always be greater than or equal to τ_S. Therefore in this situation $\mathcal{S}_i(k)$ and $\mathcal{S}_j(q)$ strongly overlap. The only other possible situation which can occur, which is shown in Figure 1c, is when $\mathcal{S}_i(k)$ is overlapped by both $\mathcal{S}_j(q-1)$ and $\mathcal{S}_j(q)$. In this case the lengths of the first and second overlapping intervals are $\tau_D - (\bar{t}_{j(q-1)} - \bar{t}_{ik})$ and $\tau_D - (\bar{t}_{ik} - \bar{t}_{jq})$ respectively and at least one of these lengths will always be greater than or equal to τ_S. Thus in this situation, $\mathcal{S}_i(k)$ strongly overlaps $\mathcal{S}_j(q-1)$ or $\mathcal{S}_j(q)$ or both. We summarize.

Lemma 1 (Overlaps) *Let i and j be distinct integers in $\{1, 2, \dots, n\}$. Let $\bar{t}_{ik}$ be fixed and define $q = \lceil \bar{t}_{ik} \rceil_j$. The only possible sensing periods of agent j which $\mathcal{S}_i(k)$ can overlap are $\mathcal{S}_j(q-1)$ and $\mathcal{S}_j(q)$; moreover $\mathcal{S}_i(k)$ must strongly overlap at least one of these. In addition,*

1. *$\mathcal{S}_i(k) \cap \mathcal{S}_j(q) \succ \tau_S$ if and only if $\bar{t}_{jq} - \bar{t}_{ik} \leq \tau_D - \tau_S$*

2. $\mathcal{S}_i(k) \cap \mathcal{S}_j(q-1) \succ \tau_S$ *if and only if* $\bar{t}_{ik} - \bar{t}_{j(q-1)} \leq \tau_D - \tau_S$.

Note that for agent j to be a registered neighbor of agent i at the beginning of agent i's *kth* maneuvering period, it is necessary and sufficient that agent j be "within range of agent i" {i.e., within agent i's sensing region} during a sensing period of agent j which strongly overlaps $\mathcal{S}_i(k)$. Consider again Figure 1 where $q = \lceil \bar{t}_{ik} \rceil_j$. In the situation depicted in Figure 1a, agent j will be a registered neighbor of agent i just in case $||x_i(\bar{t}_{ik}) - x_j(\bar{t}_{j(q-1)})|| \leq r$; moreover if this condition holds, $x_j(\bar{t}_{j(q-1)})$ will be the registered position of agent j. Similarly in the situation shown in Figure 1b, agent j will be a registered neighbor of agent i just in case $||x_j(\bar{t}_{jq}) - x_i(\bar{t}_{ik})|| \leq r$; moreover if this condition holds, $x_j(\bar{t}_{jq})$ will be the registered position of agent j. The remaining situation shown in Figure 1c is slightly more complicated. If on the one hand, the length of the second overlap is greater than or equal to τ_S and $||x_j(\bar{t}_{jq}) - x_i(\bar{t}_{ik})|| \leq r$ then agent j will be a registered neighbor of agent i with registered position $x_j(\bar{t}_{jq})$. If either of these two conditions fails to hold, if the length of the first overlap is greater than or equal to τ_S, and if $||x_i(\bar{t}_{ik}) - x_j(\bar{t}_{j(q-1)})|| \leq r$, then agent j will be a registered neighbor of agent i and $x_j(\bar{t}_{j(q-1)})$ will be its registered position. The following proposition summarizes these observations.

Proposition 1 (Neighbor Characterization) *Let* $i, j \in \{1, 2, \ldots, n\}$ *and* $\bar{t}_{ik}$ *be fixed an let* $q = \lceil \bar{t}_{ik} \rceil_j$. *Then agent* j *is a registered neighbor of agent* i *at the beginning of agent* i*'s kth maneuvering period if and only if at least one of the following is true.*

A. $\mathcal{S}_i(k) \cap \mathcal{S}_j(q) \succ \tau_S$ *and* $||x_j(\bar{t}_{jq}) - x_i(\bar{t}_{ik})|| \leq r$.

B. $\mathcal{S}_i(k) \cap \mathcal{S}_j(q-1) \succ \tau_S$ *and* $||x_i(\bar{t}_{ik}) - x_j(\bar{t}_{j(q-1)})|| \leq r$

Moreover, if A is true, then $x_j(\bar{t}_{jq})$ *is the registered position of agent* j *at the beginning of agent* i*'s kth maneuvering period and if A is not true while B is, then* $x_j(\bar{t}_{j(q-1)})$ *is the registered position of agent* j *at the beginning of agent* i*'s kth maneuvering period.*

2.1.2 Neighbor Relationship Symmetry

The definition of a registered neighbor determines a relationship between agents similar to the symmetric relationship determined by the definition of a registered neighbor in the synchronous case [5]. Suppose that agent j is a registered neighbor of agent i at the beginning of agent i's *kth* maneuvering period. In view of Proposition 1, either condition A or condition B must hold. Suppose first that condition A is true. Then $\mathcal{S}_i(k)$ strongly overlaps $\mathcal{S}_j(q)$ and agent i is in range of agent j during the overlap. There are two cases to consider. First, it is possible that $\mathcal{S}_i(k+1)$ also strongly overlaps $\mathcal{S}_j(q)$ for at least τ_S time units and agent i is in range of agent j during this overlap; in this case agent i would be a registered neighbor of agent j at time $\bar{t}_{jq}$ and $x_i(\bar{t}_{i(k+1)})$ would be its registered position. Second, it is possible that either $\mathcal{S}_i(k+1)$ does not strongly overlap $\mathcal{S}_j(q)$ or that agent i is not in range of agent j during this overlap; in this case agent i would be a registered neighbor of agent j at time $\bar{t}_{jq}$ and $x_i(\bar{t}_{ik})$ would be its registered position. Thus in summary, if condition A is true then agent i will be a registered neighbor of agent j at time $\bar{t}_{jq}$ with registered position which could be either $x_i(\bar{t}_{ik})$ or $x_i(\bar{t}_{i(k+1)})$.

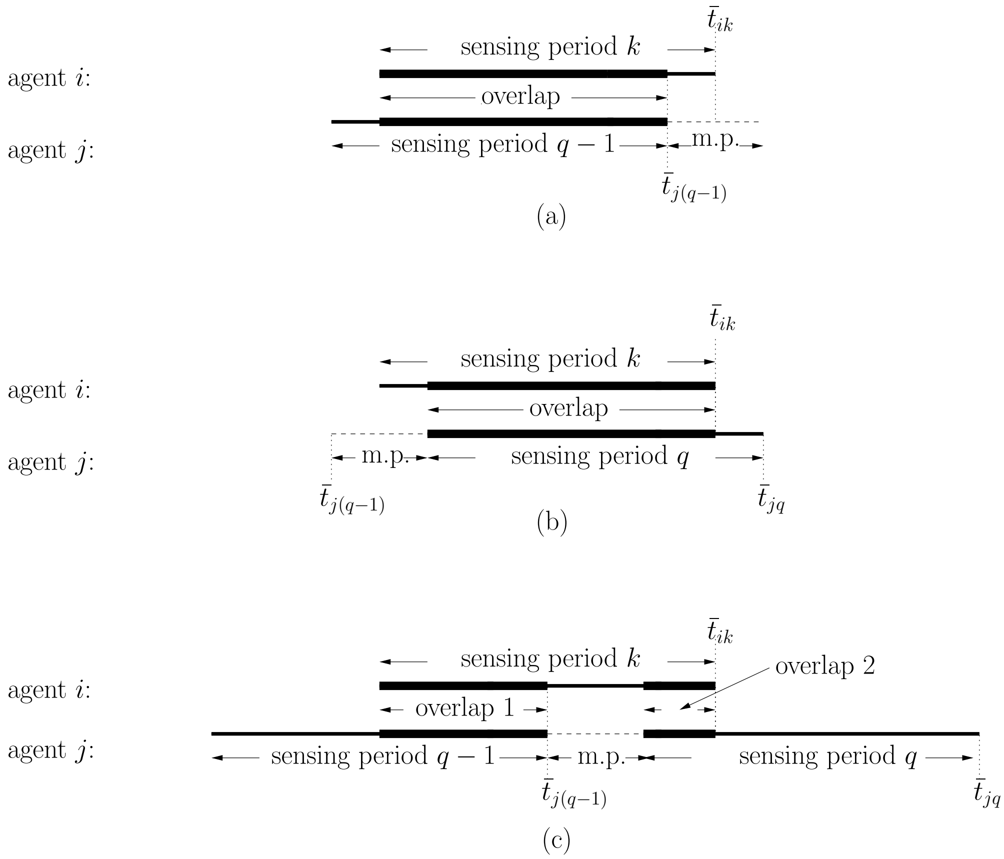

Figure 1: Sensing Period Overlaps

Suppose next that condition A does not hold. In view of Proposition 1, condition B must hold. In other words, $\mathcal{S}_i(k)$ must strongly overlap $\mathcal{S}_j(q-1)$ and agent i must be in range of agent j during this overlap. In view of Lemma 1, this must be the last sensing period of agent i with these properties because we've assumed that condition A does not hold. Therefore agent i must be a registered neighbor of agent j at time $\bar{t}_{j(q-1)}$ and $x_i(\bar{t}_{ik})$ must be its registered position. We summarize.

Proposition 2 (Neighbor Relationship Symmetry) *Suppose that agent j is a registered neighbor of agent i at the beginning of agent i's kth maneuvering period. Let $q = \lceil \bar{t}_{ik} \rceil_j$. If condition A of Proposition 1 holds, then agent i is a registered neighbor of agent j at the beginning of agent j's qth maneuvering period with either $x_i(\bar{t}_{ik})$ or $x_i(\bar{t}_{i(k+1)})$ as its registered position. If condition A of Proposition 1 does not hold, then condition B must hold and agent i is a registered neighbor of agent j at the beginning of agent j's $(q-1)$st maneuvering period with registered position $x_i(\bar{t}_{ik})$.*

2.1.3 Motion Constraint

In the synchronous case treated in [4], each agent's way points are constrained to positions defined in such a way so that no agent can lose any of its neighbors as it moves from one way point to the next. This is accomplished by adopting a clever idea proposed in [4] which we call the *pairwise motion constraint*. Neighbor retention can also be achieved in the asynchronous case by enforcing the following constraint. Agent i is said to satisfy the *motion constraints induced by its neighbors*, if for each $j \in \{1, 2, \dots, n\}$ for which $j \neq i$ and each $k \in \{1, 2, \dots\}$ for which agent j is a registered neighbor of agent i at the beginning of maneuvering period k, the position to which agent i moves at the end of the period is within a closed disk of diameter r centered at the mean of agent i's position at the beginning of the period {i.e., at time $\bar{t}_{ik}$} and the registered position of agent j at the beginning of the period. As mentioned, in the synchronous case, satisfaction of the pairwise motion constraint by agent i and neighbor j causes each to retain the other as a neighbor. The following proposition implies that essentially the same thing is true in the asynchronous case when the induced motion constraints are satisfied by agents i and j.

Proposition 3 (Neighbor Retention) *Suppose that agents i and j satisfy the motion constraints induced by their registered neighbors. If agent j is a registered neighbor of agent i at the beginning of agent i's kth maneuvering period, then agent j is also a registered neighbor of agent i at the beginning of agent i's $k+1st$ maneuvering period.*

In proving Proposition 3 and several subsequent claims we will make use of the inequalities

$$\bar{t}_{j(q+p)} - \bar{t}_{jq} \geq p\tau_D, \quad p \in \{0, 1, 2, \dots\}, \quad q \in \{1, 2, \dots\}, \quad j \in \{1, 2, \dots, n\} \tag{2}$$

and

$$\bar{t}_{i(k+1)} - \bar{t}_{ik} \leq 2(\tau_D - \tau_S), \quad k \in \{1, 2, \dots\}, \quad i \in \{1, 2, \dots, n\} \tag{3}$$

which are both direct consequences of the definitions of the sensing and maneuver periods and (1). To justify (2), let us first recall that for each integer $s \geq 1$, $\bar{t}_{js}$ is at the end of agent j's sth sensing period. In addition, agent j's sensing periods do not intersect and are each of length τ_D. It follows that $\bar{t}_{j(s+1)} - \bar{t}_{js} \geq \tau_D, \quad s \geq 1$, and thus that (2) is true. To justify (3), note that $\bar{t}_{i(k+1)}$ can be written as $\bar{t}_{i(k+1)} = \bar{t}_{ik} + \tau_D + \tau$ where τ is the length of agent i's kth maneuvering period. Since τ is constrained to satisfy $\tau \leq \tau_{M_i}$, we can write $\bar{t}_{i(k+1)} \leq \bar{t}_{ik} + \tau_D + \tau_{M_i}$. From this and (1) it follows that $\bar{t}_{i(k+1)} \leq \bar{t}_{ik} + \tau_D + (\tau_D - 2\tau_S)$ and thus that (3) is true.

To prove Proposition 3, we will make use of the two conditions characterizing a registered neighbor in Proposition 1. Each of these conditions in turn involves both an overlap requirement and a range requirement. The next lemma provides the needed facts about the way in which two agents sensing periods overlap. This is followed by Lemma 3 which provides the range information needed to prove Proposition 3 and subsequent claims.

Lemma 2 *Let i and j be distinct integers in $\{1, 2, \dots, n\}$. Let $\bar{t}_{ik}$ be fixed and define $q = \lceil \bar{t}_{ik} \rceil_j$. Then*

$$\lceil \bar{t}_{i(k+1)} \rceil_j \in \{q, q+1, q+2\} \tag{4}$$

1. *If $\lceil \bar{t}_{i(k+1)} \rceil_j = q$, then $\mathcal{S}_i(k+1) \cap \mathcal{S}_j(q) \succ \tau_S$.*
2. *If $\lceil \bar{t}_{i(k+1)} \rceil_j = q+1$, then $\mathcal{S}_i(k+1) \cap \mathcal{S}_j(q) \succ \tau_S$ or $\mathcal{S}_i(k+1) \cap \mathcal{S}_j(q+1) \succ \tau_S$.*
3. *If $\lceil \bar{t}_{i(k+1)} \rceil_j = q+2$, then $\mathcal{S}_i(k) \cap \mathcal{S}_j(q) \succ \tau_S$ and $\mathcal{S}_i(k+1) \cap \mathcal{S}_j(q+1) \succ \tau_S$.*

Moreover, if $\lceil \bar{t}_{i(k+1)} \rceil_j \in \{q+1, q+2\}$, then $\mathcal{S}_i(k)$ and $\mathcal{S}_i(k+1)$ are the only sensing periods of agent i which can strongly overlap $\mathcal{S}_j(q)$.

Proof of Lemma 2: It will be shown first that (4) is true. Since $\bar{t}_{ik} \in (\bar{t}_{j(q-1)} - \bar{t}_{jq}]$ and $\bar{t}_{i(k+1)} > \bar{t}_{ik}$, it must be true that $\bar{t}_{i(k+1)} > \bar{t}_{j(q-1)}$. Thus $\lceil \bar{t}_{i(k+1)} \rceil \geq q$. To prove that $\lceil \bar{t}_{i(k+1)} \rceil_j \leq q+2$, we use (3) and the fact that $\bar{t}_{ik} \leq \bar{t}_{jq}$ to write $\bar{t}_{i(k+1)} \leq 2(\tau_D - \tau_S) + \bar{t}_{jq}$. In view of (2) {with $p = 1$}, $2(\tau_D - \tau_S) + \bar{t}_{jq} \leq \tau_D + \bar{t}_{j(q+1)} \leq \bar{t}_{j(q+2)}$. Therefore $\bar{t}_{i(k+1)} \leq \bar{t}_{j(q+2)}$. This means that $\lceil \bar{t}_{i(k+1)} \rceil_j \leq q+2$. Thus (4) is true.

To prove assertion 1, we use (2) with i substituted for j and $p = 1$ to write $\bar{t}_{i(k+1)} \geq \bar{t}_{ik} + \tau_D$. In view of the definition of q, $\bar{t}_{ik} > \bar{t}_{j(q-1)}$. Therefore $\bar{t}_{i(k+1)} - \bar{t}_{j(q-1)} > \tau_D > \tau_D - \tau_S$. The hypothesis $\lceil \bar{t}_{i(k+1)} \rceil = q$ implies that Lemma 1 holds with $k+1$ substituted for k. Thus $\mathcal{S}_i(k+1)$ and $\mathcal{S}_j(q-1)$ cannot overlap because of the lemma's last claim. Since the lemma also states that $\mathcal{S}_i(k+1)$ must strongly overlap either $\mathcal{S}_j(q-1)$ or $\mathcal{S}_j(q)$, it must be true that $\mathcal{S}_i(k+1)$ strongly overlaps $\mathcal{S}_j(q)$. Therefore assertion 1 is true.

Assertion 2 assumes that $\lceil \bar{t}_{i(k+1)} \rceil_j = q+1$. Lemma 1 thus applies with $k+1$ and $q+1$ replacing k and q respectively. From this it follows that the only sensing periods of agent j which can overlap $\mathcal{S}_1(k+1)$ are $\mathcal{S}_j(q)$ and $\mathcal{S}_j(q+1)$; moreover $\mathcal{S}_1(k+1)$ must overlap strongly overlap at least one of these. Thus assertion 2 is true.

Assertion 3 assumes that $\lceil \bar{t}_{i(k+1)} \rceil_j = q+2$. Thus $\bar{t}_{j(q+1)} < \bar{t}_{i(k+1)}$. But $\bar{t}_{jq} + \tau_D \leq \bar{t}_{j(q+1)}$ because of (2) {with $p = 1$} and $\bar{t}_{i(k+1)} \leq \bar{t}_{ik} + 2(\tau_D - \tau_S)$ because of (3). Therefore $\bar{t}_{jq} \leq \bar{t}_{ik} + \tau_D - 2\tau_S$. It follows that $\bar{t}_{jq} - \bar{t}_{ik} + \tau_D - \tau_S$. Therefore by the first assertion of Lemma (1), $\mathcal{S}_i(k)$ and $\mathcal{S}_j(q)$ must strongly overlap. It remains to be shown that $\mathcal{S}_i(k+1) \cap \mathcal{S}_j(q+1) \succ \tau_S$ if $\lceil \bar{t}_{i(k+1)} \rceil_j = q+2$. Since $\lceil \bar{t}_{i(k+1)} \rceil_j = q+2$, Lemma 1 applies with $k+1$ and $q+2$ replacing k and q respectively. Thus prove that $\mathcal{S}_i(k+1)$ and $\mathcal{S}_j(q+1)$ also strongly overlap, it is enough to show that $\bar{t}_{i(k+1)} - \bar{t}_{j(q+1)} \leq \tau_D - \tau_S$. To do this, we first use (3) and the fact that $\bar{t}_{ik} \leq \bar{t}_{jq}$ to write $\bar{t}_{i(k+1)} \leq \bar{t}_{jq} + 2(\tau_D - \tau_S)$. From this and (2) with $p = 1$ there follows $\bar{t}_{i(k+1)} \leq \bar{t}_{j(q+1)} + \tau_D - 2\tau_S$. Therefore $\bar{t}_{i(k+1)} \leq \bar{t}_{j(q+1)} + \tau_D - \tau_S$. Thus $\mathcal{S}_i(k+1) \cap \mathcal{S}_j(q+1) \succ \tau_S$ so assertion 3 is true.

Now suppose that $\lceil \bar{t}_{i(k+1)} \rceil_j \in \{q+1, q+2\}$. Then in either case $\bar{t}_{jq} \leq \bar{t}_{i(k+1)}$. Therefore $\bar{t}_{ik} \leq \bar{t}_{jq} \leq \bar{t}_{i(k+1)}$. If $\bar{t}_{ik} \neq \bar{t}_{jq}$ then $\bar{t}_{ik} < \bar{t}_{jq} \leq \bar{t}_{i(k+1)}$ which means that $\lceil \bar{t}_{jq} \rceil = \bar{t}_{i(k+1)}$; thus Lemma 1 applies with k an q replaced by q and $k+1$ respectively. Therefore in this case $\mathcal{S}_i(k)$ and $\mathcal{S}_i(k+1)$ are the only sensing periods of agent i which can strongly overlap $\mathcal{S}_j(q)$. Now suppose that $\bar{t}_{ik} = \bar{t}_{jq}$. This means that $\lceil \bar{t}_{jq} \rceil = \bar{t}_{ik}$; thus Lemma 1 applies with k an q interchanged. Therefore in this case $\mathcal{S}_i(k-1)$ and $\mathcal{S}_i(k)$ are the only sensing periods of agent i which can strongly overlap $\mathcal{S}_j(q)$. To complete the proof, it is enough to show that $\mathcal{S}_i(k-1)$ cannot strongly overlap $\mathcal{S}_j(q)$. Towards this end, first note that $\bar{t}_{ik} \geq \bar{t}_{i(k-1)} + \tau_D$ because of (2). Thus $\bar{t}_{jq} \geq \bar{t}_{i(k-1)} + \tau_D$ so $\bar{t}_{jq} - \bar{t}_{i(k-1)} > +\tau_D - \tau_S$. Therefore $\mathcal{S}_i(k-1)$ cannot strongly overlap $\mathcal{S}_j(q)$ because of Lemma 1. ■

Lemma 3 *Let $q = \lceil \bar{t}_{ik} \rceil_j$. Suppose that agents i and j satisfy the motion constraints induced by their registered neighbors. If agent j is a registered neighbor of agent i at the beginning of agent i's kth maneuvering period, then*

$$||x_i(\bar{t}_{i(k+1)}) - x_j(\bar{t}_{jq^*})|| \leq r \quad (5)$$

$$||x_i(\bar{t}_{i(k+1)}) - x_j(\bar{t}_{j(q^*+1)})|| \leq r \quad (6)$$

where $q^* = q$ if condition A of Proposition 1 is true and $q^* = q-1$ if it is not. Moreover, in either *case*

$$||x_i(\bar{t}_{ik}) - x_j(\bar{t}_{jq})|| \leq r \tag{7}$$

Proof of Lemma 3: First suppose that agent j is a registered neighbor of agent i at the beginning of maneuvering period k. Thus by Proposition 1, x_{jq^*} is agent j's registered position and

$$||x_i(\bar{t}_{ik}) - x_j(\bar{t}_{jq^*})|| \leq r \tag{8}$$

where $q^* = q$ if condition A holds and $q^* = q-1$ if it does not. The positions of agent i at the beginning and end of its kth maneuvering period are $x_i(\bar{t}_{ik})$ and $x_i(t_{ik})$ respectively. Therefore since agent i satisfies the motion constraint induced by agent j during this period, $||x_i(t_{ik}) - \frac{1}{2}\{x_i(\bar{t}_{ik}) + x_j(\bar{t}_{jq^*})\}|| \leq \frac{r}{2}$. But $x_i(\bar{t}_{i(k+1)}) = x_i(t_{ik})$ because agent i does not move during sensing period $[t_{ik},\ \bar{t}_{i(k+1)})$. This enables us to re-write the preceding inequality as

$$||x_i(\bar{t}_{i(k+1)}) - \frac{1}{2}\{x_i(\bar{t}_{ik}) + x_j(\bar{t}_{jq^*})\}|| \leq \frac{r}{2} \tag{9}$$

Observe that

$$x_i(\bar{t}_{i(k+1)}) - x_j(\bar{t}_{jq^*}) \quad = \quad x_i(\bar{t}_{i(k+1)}) - \frac{1}{2}\{x_i(\bar{t}_{ik}) + x_j(\bar{t}_{jq^*})\} - \frac{1}{2}\left(x_j(\bar{t}_{jq^*}) - x_i(\bar{t}_{ik})\right)$$

Hence

$$||x_i(\bar{t}_{i(k+1)}) - x_j(\bar{t}_{jq^*})|| \quad \leq \quad \left\|x_i(\bar{t}_{i(k+1)}) - \frac{1}{2}\{x_i(\bar{t}_{ik}) + x_j(\bar{t}_{jq^*})\}\right\| + \left\|\frac{1}{2}\left(x_j(\bar{t}_{jk^*}) - x_i(\bar{t}_{ik})\right)\right\|$$

From this (8) and (9) there follows $||x_i(\bar{t}_{i(k+1)}) - x_j(\bar{t}_{jq^*})|| \leq \frac{r}{2} + \frac{r}{2} = r$. Therefore (5) is true.

It will now be shown that (6) is also true. By Proposition 2, agent i is a registered neighbor of agent j at the beginning of agent j's q^*th maneuvering period where $q^* = q$ if condition A of Proposition 1 holds and $q^* = q - 1$ if it does not. Thus by Proposition 1

$$||x_j(\bar{t}_{jq^*}) - \bar{x}_i|| \leq r \tag{10}$$

where $\bar{x}_i$ denotes the registered position of agent i at $\bar{t}_{jq^*}$. The positions of agent j at the beginning and end of its q^*th maneuvering period are $x_j(\bar{t}_{jq^*})$ and $x_j(t_{jq^*})$ respectively. Therefore since agent j satisfies the motion constraint induced by agent i during this period, $||x_j(t_{jq^*}) - \frac{1}{2}\{x_j(\bar{t}_{jq^*}) + \bar{x}_i\}|| \leq \frac{r}{2}$. But $x_j(\bar{t}_{j(q^*+1)}) = x_j(t_{jq^*})$ because agent j does not move during sensing period $q^* + 1$. Therefore

$$||x_j(\bar{t}_{j(q^*+1)}) - \frac{1}{2}\{x_j(\bar{t}_{jq^*}) + \bar{x}_i\}|| \leq \frac{r}{2} \tag{11}$$

In view of Proposition 2, $\bar{x}_i$ could be either $x_i(\bar{t}_{ik})$ or $x_i(\bar{t}_{i(k+1)})$ if condition A of Proposition 1 hold while $\bar{x}_i = x_i(\bar{t}_{ik})$ if it does not. Consider first the case when $\bar{x}_i = x_i(\bar{t}_{ik})$. It is then possible to re-write (11) as

$$||x_j(\bar{t}_{j(q^*+1)}) - \frac{1}{2}\{x_j(\bar{t}_{jq^*}) + x_i(\bar{t}_{ik}\}|| \leq \frac{r}{2} \tag{12}$$

But

$$\begin{aligned}||x_i(\bar{t}_{i(k+1)}) - x_j(\bar{t}_{j(q^*+1)})|| &= ||x_i(\bar{t}_{i(k+1)}) - \frac{1}{2}\{x_i(\bar{t}_{ik}) + x_j(\bar{t}_{jq^*})\} - (x_j(\bar{t}_{j(q^*+1)}) \\ &\quad -\frac{1}{2}\{x_j(\bar{t}_{jq^*}) + x_i(\bar{t}_{ik})\})|| \\ &\leq ||x_i(\bar{t}_{i(k+1)}) - \frac{1}{2}\{x_i(\bar{t}_{ik}) + x_j(\bar{t}_{jq^*})\}|| \\ &\quad +||x_j(\bar{t}_{j(q^*+1)}) - \frac{1}{2}\{x_j(\bar{t}_{jq^*}) + x_i(\bar{t}_{ik})\}||\end{aligned}$$

From this, (9) and (12) it follows that $||x_i(\bar{t}_{i(k+1)}) - x_j(\bar{t}_{j(q^*+1)})|| \leq r$ and thus that (6) holds.

It will now be shown that (6) also holds for the case when $\bar{x}_i = x_i(\bar{t}_{i(k+1)})$ which only occurs when $q^* = q$. Assuming this possibility (11) can be written as

$$||x_j(\bar{t}_{j(q^*+1)}) - \frac{1}{2}\{x_j(\bar{t}_{jq^*}) + x_i(\bar{t}_{i(k+1)})\}|| \leq \frac{r}{2} \tag{13}$$

Observe that it is possible to write

$$\begin{aligned}x_j(\bar{t}_{j(q^*+1)}) - x_i(\bar{t}_{i(k+1)}) &= x_j(\bar{t}_{j(q^*+1)}) - \frac{1}{2}\{x_j(\bar{t}_{jq^*}) + x_i(\bar{t}_{i(k+1)})\} \\ &\quad -\frac{1}{2}\left(x_i(\bar{t}_{i(k+1)}) - \frac{1}{2}\{x_j(\bar{t}_{jq^*}) + x_i(\bar{t}_{ik})\}\right) + \frac{1}{4}\left(x_j(\bar{t}_{jq^*}) - x_i(\bar{t}_{ik})\right)\end{aligned}$$

Clearly

$$\begin{aligned}||x_j(\bar{t}_{j(q^*+1)}) - x_i(\bar{t}_{i(k+1)})|| &\leq \left\|x_j(\bar{t}_{j(q^*+1)}) - \frac{1}{2}\{x_j(\bar{t}_{jq^*}) + x_i(\bar{t}_{i(k+1)})\}\right\| \\ &\quad +\frac{1}{2}\left\|x_i(\bar{t}_{i(k+1)}) - \frac{1}{2}\{x_j(\bar{t}_{jq^*}) + x_i(\bar{t}_{ik})\}\right\| + \frac{1}{4}\left\|x_j(\bar{t}_{jq^*}) - x_i(\bar{t}_{ik})\right\|\end{aligned}$$

Using (8), (9) and (13) we thus obtain $||x_j(\bar{t}_{j(q^*+1)}) - x_i(\bar{t}_{i(k+1)})|| \leq \frac{r}{2} + \frac{r}{4} + \frac{r}{4} = r$. Thus (6) holds in this case too.

In view of (8), (7) is true if $q^* = q$. To prove that (7) also holds if $q^* = q - 1$, we first write

$$x_j(\bar{t}_{j(q^*+1)}) - x_i(\bar{t}_{ik}) = x_j(\bar{t}_{j(q^*+1)}) - \frac{1}{2}\{x_j(\bar{t}_{jq^*}) + x_i(\bar{t}_{ik})\} - \frac{1}{2}\left(x_i(\bar{t}_{ik}) - x_j(\bar{t}_{jq^*})\right)$$

Therefore

$$||x_j(\bar{t}_{j(q^*+1)}) - x_i(\bar{t}_{ik})|| \leq \left\|x_j(\bar{t}_{j(q^*+1)}) - \frac{1}{2}\{x_j(\bar{t}_{jq^*}) + x_i(\bar{t}_{ik})\}\right\| + \left\|\frac{1}{2}\left(x_i(\bar{t}_{ik}) - x_j(\bar{t}_{jq^*})\right)\right\| \tag{14}$$

But if $q^* = q-1$, both (8) and (12) hold. From these inequalities and (14) it follows that $||x_j(\bar{t}_{j(q^*+1)}) - x_i(\bar{t}_{ik})|| \leq \frac{1}{r} + \frac{1}{r} = r$ and therefore that (7) is true. ■

Proof of Proposition 3: Consider first the case when $\lceil \bar{t}_{i(k+1)} \rceil = q$. If condition A of Lemma 1 holds, then $q^* = q$ so does

$$||x_i(\bar{t}_{i(k+1)}) - x_j(\bar{t}_{jq})|| \leq r \tag{15}$$

because of (5). On the other hand, if condition A of Lemma 1 does not hold, then $q = q^* - 1$ and (15) still holds, in this case because of (6). Since $\lceil \bar{t}_{i(k+1)} \rceil = q$, it must be true that $\mathcal{S}_i(k+1) \cap \mathcal{S}_j(q) \succ \tau_S$

because of Lemma 2. This and (15) mean that condition A of Proposition 1 is satisfied with $k+1$ substituted for k. Therefore agent j is a registered neighbor of agent i at $\bar{t}_{i(k+1)}$.

Now suppose that $\lceil \bar{t}_{i(k+1)} \rceil \in \{q+1, q+2\}$. Consider first the case when condition A of Proposition 1 holds. Then Lemma 3 applies with $q^* = q$, so

$$||x_i(\bar{t}_{i(k+1)}) - x_j(\bar{t}_{j(q+1)})|| \leq r \tag{16}$$

and

$$||x_i(\bar{t}_{i(k+1)}) - x_j(\bar{t}_{jq})|| \leq r \tag{17}$$

If $\lceil \bar{t}_{i(k+1)} \rceil = q+1$, then $\mathcal{S}_i(k+1)$ must strongly overlap either $\mathcal{S}_j(q)$ or $\mathcal{S}_j(q+1)$ because of Lemma 2. In view of (16) and (17), condition A of Proposition 1 is satisfied in either situation with $k+1$ substituted for k and $q+1$ substituted for q. Therefore agent j is a registered neighbor of agent i at $\bar{t}_{i(k+1)}$. If $\lceil \bar{t}_{i(k+1)} \rceil = q+2$, then $\mathcal{S}_i(k+1)$ and $\mathcal{S}_j(q+1)$ still must strongly overlap because of Lemma 2. Thus in this case condition B of Proposition 1 is satisfied with $k+1$ substituted for k and $q+2$ substituted for q. Therefore agent j is a registered neighbor of agent i at $\bar{t}_{i(k+1)}$.

Consider finally the case when condition A of Proposition 1 does not hold. Since (7) hold, $\mathcal{S}_i(k)$ and $\mathcal{S}_j(q)$ cannot overlap. Therefore $\lceil t_{ik} \rceil \neq q+2$ because of statement 3 in Lemma 2. Thus $\lceil t_{ik} \rceil = q+1$. In addition, Lemma 2 states that the only sensing periods of agent i which can strongly overlap $\mathcal{S}_j(q)$ are $\mathcal{S}_i(k)$ and $\mathcal{S}_i(k+1)$. Since $\mathcal{S}_j(q)$ must be strongly overlap at least one sensing period of agent i, it must be true that

$$\mathcal{S}_j(q) \cap \mathcal{S}_i(k+1) \succ \tau_S \tag{18}$$

Since condition A of Proposition 1 does not hold condition B must hold, because agent j is a neighbor of agent i at $\bar{t}_{ik}$. Thus Lemma 3 applies with $q^* = q-1$ so by (6)

$$||x_i(\bar{t}_{i(k+1)}) - x_j(\bar{t}_{jq})|| \leq r \tag{19}$$

Since $\lceil \bar{t}_{i(k+1)} \rceil = q+1$, (19) and (18) show that condition B of Proposition 1 is satisfied with $k+1$ and $q+1$ substituted for k and q respectively. ■

2.2 Unsynchronized Agent strategies

We are interested in strategies which cause agents to retain their registered neighbors. We therefore make the following assumption.

Cooperation Assumption: *Each agent i satisfies the motion constraints induced by each of its registered neighbors.*

Suppose that the cooperation assumption is satisfied. Proposition 3 states that if agent j is a registered neighbor of agent i during maneuvering interval k then it will also be a registered neighbor of agent i during maneuvering interval $k+1$. In other words, if the cooperation assumption is satisfied, each agent retains all of its prior registered neighbors as the system evolves. Thus if $\mathcal{N}_i(k)$ denotes the set of labels of agent i's neighbors at the beginning of its kth maneuvering period, then $\mathcal{N}_i(k) \subset \mathcal{N}_i(k+1),\ k \geq 1$.

Agent i's *kth way-point* $\bar{x}_i(k)$ is the point to which agent i moves at the end of its kth maneuvering period. Thus if $x_i(t)$ denotes the position of agent i at time t represented in a world coordinate system, then $x_i(t_{ik})$ and agent i's kth way-point are one and the same. The rule which determines $\bar{x}_i(k)$ is essentially the same as considered previously for the synchronous case in [4, 5], except that now $\bar{x}_i(k)$ depend on agent i's its own position at the beginning of its kth maneuvering period and the registered {relative} positions of agent i's registered neighbors at the beginning of the period. In particular if agent i has m_{ik} registered neighbors at time $\bar{t}_{ik}$ with registered positions $z_1, z_2, \ldots, z_{m_{ik}}$ relative to agent i's, then agent i moves to the position $\bar{x}_i(k) = x_i(t_{i(k-1)}) + u_{m_{ik}}(z_1, \ldots, z_{m_{ik}})$ at the end of the period where

$$z_j = x_{ii_j}(\bar{t}_{ik}) - x_i(t_{i(k-1)}), \quad j \in \{1, 2, \ldots, m_{ik}\}, \tag{20}$$

and $x_{ii_j}(\bar{t}_{ik})$ is the registered position of neighbor i_j at time $\bar{t}_{ik}$. As in [5], $u_0 = 0$ and for $m \in \{1, \ldots, n-1\}$ u_m is a continuous control law mapping $\mathbb{D}^m$ into $\mathbb{D}_M$ where $\mathbb{D}$ and $\mathbb{D}_M$ are the closed disks of radii r and r_M respectively, centered at the origin in $\mathbb{R}^2$. For $m > 0$, u_m is defined so that the aforementioned neighbor motion constraint is satisfied and, in addition so that for each $\{z_1, z_2, \ldots, z_m\} \in \mathbb{D}^m$, $u_m(z_1, z_2, \ldots, z_n)$ is in the convex hull of $\{0, z_1, z_2, \ldots, z_m\}$, but not at a corner unless $z_1 = z_2 = \cdots = z_m = 0$. Examples of $u_m(\cdot)$ satisfying these *control law requirements* can be found in [4, 5].

Since each agent is assumed to move to its kth way point at the end of its kth maneuvering period, agent i's position at time t_{ik} is given by

$$x_i(t_{ik}) = x_i(t_{i(k-1)}) + u_{m_{ik}}(x_{ii_1}(\bar{t}_{ik}) - x_i(t_{i(k-1)}), \ldots,\ x_{ii_{m_{ik}}}(\bar{t}_{ik}) - x_i(t_{i(k-1)})) \tag{21}$$

In view of Proposition 1 and (7), the formulas for the $x_{ij}(\bar{t}_{ik})$ can be written as

$$x_{ij}(\bar{t}_{ik}) = \left\{ \begin{array}{ll} x_j(\bar{t}_{jq}) & \text{if } \mathcal{S}_i(k) \cap \mathcal{S}_j(q) \succ \tau_S \\ x_j(\bar{t}_{j(q-1)}) & \text{otherwise} \end{array} \right\}, \quad j \in \mathcal{N}_i(k) \tag{22}$$

where $q = \lceil \bar{t}_{ik} \rceil_j$ and

$$\begin{aligned} \mathcal{N}_i(k) \;=\; & \{j : ||x_i(\bar{t}_{ik}) - x_j(\bar{t}_{iq})|| \leq r \text{ and } \mathcal{S}_i(k) \cap \mathcal{S}_j(q) \succ \tau_S\} \bigcup \{j : ||x_i(\bar{t}_{ik}) - x_j(\bar{t}_{i(q-1)})|| \leq r \\ & \text{and } \mathcal{S}_i(k) \cap \mathcal{S}_j(q-1) \succ \tau_S\} \end{aligned} \tag{23}$$

The expressions for the $x_{ij}(\bar{t}_{ik})$ in (22) are a direct consequence of the characterization of registered positions in Proposition 1, the fact that (7) holds whenever $j \in \mathcal{N}_i(k)$, and the implication of Lemma 1 that $\mathcal{S}_i(k) \cap \mathcal{S}_j(q-1) \succ \tau_S$ whenever $\mathcal{S}_i(k) \cap \mathcal{S}_j(q) \not\succ \tau_S$. Of course the neighbor set $\mathcal{N}_i(k)$ and the registration positions $x_{ij},\ j \in \mathcal{N}_i(k)$ all depend on i and k.

3 Main Results

Note that because agents do not move during sensing periods, for each $i \in \{1, 2, \ldots, n\}$ the positions of agent i at times $t_{i(k-1)}$ and t_{ik} are the same as at times $\bar{t}_{ik}$ and $\bar{t}_{i(k+1)}$ respectively. Thus (21) can also be written as

$$x_i(\bar{t}_{i(k+1)}) = x_i(\bar{t}_{ik}) + u_{m_{ik}}(x_{ii_1}(\bar{t}_{ik}) - x_i(\bar{t}_{ik}), \ldots,\ x_{ii_{m_{ik}}}(\bar{t}_{ik}) - x_i(\bar{t}_{ik})) \tag{24}$$

The n equations given by (24) for $i \in \{1, 2, \ldots, n\}$ together with (22) and (23) completely describes the evolution of the positions of the n agents under consideration as each maneuvers from way-point to way-point. Just as in the synchronous case, the analysis of these equations depends on the relationships between registered neighbors and how these relationships evolve with time. To characterize these relationships, we first extend the domain of definition of each agent's registered neighbors from its set of maneuvering period start times to a suitably defined set of "event times" common to all n agents. By an *event time* is meant any time $\bar{t}_{ik}$ at which any maneuvering period $[\bar{t}_{ik}, t_{ik})$ of any agent begins. Let $\{\bar{t}_{ik} : \ i \in \{1, 2, \ldots, n\}, \ k \geq 1\}$ denote the set of all distinct event times. Label this set's elements as $t_1, t_2, \cdots, t_p, \cdots$ in such a way so that $t_p < t_{p+1}, \ j \in \{1, 2, \ldots\}$. For $i \in \{1, 2, \ldots, n\}$, let P_i denote that strictly monotone function from the set of positive integers $\mathcal{I}$ to $\mathcal{I}$ which assigns to $k \in \mathcal{I}$ that value of $p \in \mathcal{I}$ for which $t_p = \bar{t}_{ik}$. Thus with this notation, $t_{P_i(k)} = \bar{t}_{ik}$ so agent i's registered neighbors at its kth event time $t_{P_i(k)}$, are its registered neighbors at time $\bar{t}_{ik}$. For each $i \in \{1, 2, \ldots, n\}$ we extend the domain of definition of agent i's registered neighbors from the set $\{t_{P_i(k)} : k \geq 1\}$ to the set $\{t_p : p \geq P_i(1)\}$ by stipulating that for values of t_p which are between two successive event times of agent i, say between t_{ik} and $t_{i(k+1)}$, agent i's registered neighbors are the same as its registered neighbors at time t_{ik}.

Let $\mathcal{T} \stackrel{\Delta}{=} \{t_{\bar{p}}, t_{\bar{p}+1}, t_{\bar{p}+2} \ldots\}$ denote the set of all event times greater than or equal to $t_{\bar{p}}$ where $\bar{p} \stackrel{\Delta}{=} \max\{P_1(1), P_2(1), \ldots P_n(1)\}$. Note that the registered neighbors of each agent are defined at each time t_p in $\mathcal{T}$. For each $p \geq \bar{p}$, it is therefore possible to describe neighbor relationships using a directed[1] graph $\mathbb{G}_p$ with vertex set $\{1, 2, \ldots, n\}$ and directed edge set defined so that (i, j) is a directed edge from vertex i to vertex j just in case agent j is a registered neighbor of agent i at event time t_p.

Let us partially order the set of all directed graphs with vertex set $\{1, 2, \ldots, n\}$ by agreeing to say that $\mathbb{G}$ is contained in $\bar{\mathbb{G}}$ if the edge set of $\mathbb{G}$ is a subset on the edge set of $\bar{\mathbb{G}}$. It is natural then to define the *union* of a collection of such graphs to be the directed graph with vertex set $\{1, 2, \ldots, n\}$, and edge set equaling the union of the edge sets of all of the graphs in the collection. Because of the cooperation assumption and Proposition 3, we know that each agent keeps all of its registered neighbors as the system evolves. What this means is the sequence of graphs $\mathbb{G}_{\bar{p}}, \mathbb{G}_{\bar{p}+1}, \ldots, \mathbb{G}_p, \ldots$ forms the ascending chain

$$\mathbb{G}_{\bar{p}} \subset \mathbb{G}_{\bar{p}+1} \subset \cdots \mathbb{G}_p \cdots \tag{25}$$

Because the set of directed graphs on vertices $\{1, 2, \ldots, n\}$ is a finite set, the chain must converge to the graph

$$\mathbb{G} \stackrel{\Delta}{=} \bigcup_{p=\bar{p}}^{\infty} \mathbb{G}_p \tag{26}$$

in a finite number of steps. More is true. Suppose that agent i has agent j as a registered neighbor at the beginning of one of agent i's maneuvering periods. Then because of Proposition 2, agent i must be a registered neighbor of agent j at the beginning of one of agent j's maneuvering periods. These observations together with the cooperation assumption imply that agents i and j must both eventually become and remain registered neighbors of each other. As a consequence, there must be directed arcs in $\mathbb{G}$ from vertex i to vertex j as well as from vertex j to vertex i. Clearly $\mathbb{G}$ must be a directed graph with the property that for each distinct pair of vertices - say i and j - either there is no directed arc connecting one to the other or there are two directed arcs one from vertex i to vertex j and the other from vertex j to vertex i. Directed graphs with this property are usually regarded as simple graphs whose edges represent such pairs of directed arcs [6]. In the sequel we shall adopt this viewpoint and refer to $\mathbb{G}$ as a simple graph. Our main result is as follows.

[1]It will soon be clear that the aforementioned symmetry of the neighbor relationship will ultimately enable us to characterize neighbor relationships with a simple, undirected graph as in the synchronous case.

Theorem 1 *Let* $u_0 = 0 \in \mathbb{D}_M$ *and for each* $m \in \{1, 2, \ldots, n-1\}$*, let* $u_m : \mathbb{D}^m \to \mathbb{D}_M$ *be any continuous function satisfying the aforementioned control law requirements. For each set of initial agent positions* $x_1(0), x_2(0), \ldots, x_n(0)$*, each agent's position* $x_i(t)$ *converges to a unique point* $\pi_i \in \mathbb{R}^2$ *such that for each* $i, j \in \{1, 2, \ldots, n\}$*, either* $\pi_i = \pi_j$ *or* $||\pi_i - \pi_j|| > r$*. Moreover, if agent* j *is a registered neighbor of agent* i *at the beginning of one of agent* i*'s maneuvering periods, then* $\pi_i = \pi_j$*.*

This theorem will be proved in §4.

Theorem 1 states that the strategies under consideration cause all agents' positions to converge to points in the plane with the property that each pair of such points are either equal to each other, or separated by a distance greater than r units. The theorem further states that if one agent is ever a registered neighbor of another, then both converge to the same point. Thus all n agents position will converge to a single point if any one directed graph in the ascending chain is strongly connected. We are led to the following corollary.

Corollary 1 *If at any event time* $t_p \geq t_{\bar{p}}$*, the directed graph* $\mathbb{G}_p$ *characterizing registered neighbors is strongly connected, then positions of all* n *agents converge to a common point in the plane.*

4 Analysis

The aim of this section is to establish the correctness of Theorem 1. This requires the analysis of the asymptotic behavior of the *asynchronous* process described by (22) and (24) for $i \in \{1, 2, \ldots, n\}$. Despite the apparent complexity of this process, it is possible to capture its salient features for t_s sufficiently large using a suitably defined *synchronous* discrete-time, hybrid dynamical system $\mathbb{S}$. The process of constructing a synchronous process to model the behavior of an asynchronous process is called *analytic synchronization* and has been outlined in the introduction to this paper. of interest in its own right. In the sequel we demonstrate the utility of this idea by applying it to the problem at hand.

4.1 A Synchronous Model of the Asynchronous Agent System

It is sufficient to analyze the behavior of the n agent system for times beyond the time at which each agent's neighbor set stops changing. Analytic synchronization would thus have us define $\mathbb{S}$ to be a synchronous system evolving on the event time set $\{t_p : p \in \mathcal{P}\}$ where $\mathcal{P} = \{p; p \geq p^*\}$ and p^* is the smallest values of $p \geq \bar{p}$ for which the ascending chain shown in (25) has converged to the limit graph $\mathbb{G}$ in (26). To reduce clutter we will instead define $\mathbb{S}$ to be a synchronous discrete-time dynamical system evolving on the index set $\mathcal{P}$. Thus for $p \in \mathcal{P}$, the registered neighbors of each agent do not change. For simplicity, we will only deal with the case when each agent has at least one neighbor for $t_p \geq t_{p^*}$. The position update equation (24) for agent i can thus be written as

$$x_i(\bar{t}_{i(k+1)}) = x_i(\bar{t}_{ik}) + u_{m_i}(x_{ii_1}(\bar{t}_{ik}) - x_i(\bar{t}_{ik}), \ldots, \; x_{ii_{m_i}}(\bar{t}_{ik}) - x_i(\bar{t}_{ik})) \qquad (27)$$

where m_i is a positive number and $\mathcal{N}_i \stackrel{\Delta}{=} \{i_1, i_2, \dots, i_{m_i}\}$ is the set of indices labelling agent i's registered neighbors. Just as before,

$$x_{ij}(\bar{t}_{ik}) = \begin{cases} x_j(\bar{t}_{jq}) & \text{if } \mathcal{S}_i(k) \cap \mathcal{S}_j(q) \succ \tau_S \\ x_j(\bar{t}_{j(q-1)}) & \text{otherwise} \end{cases} \tag{28}$$

and

$$\mathcal{N}_i = \{j : ||x_i(\bar{t}_{ik}) - x_j(\bar{t}_{iq})|| \leq r \text{ and } \mathcal{S}_i(k) \cap \mathcal{S}_j(q) \succ \tau_S\} \bigcup \{j : ||x_i(\bar{t}_{ik}) - x_j(\bar{t}_{i(q-1)})|| \leq r \text{ and } \mathcal{S}_i(k) \cap \mathcal{S}_j(q-1) \succ \tau_S\} \tag{29}$$

where $q = \lceil \bar{t}_{ik} \rceil_j$. Note that it must be true that

$$||x_j(\bar{t}_{jq}) - x_i(\bar{t}_{ik})|| \leq r \tag{30}$$

because of (7). In view of (29) it also must be true that

$$||x_j(\bar{t}_{j(q-1)}) - x_i(\bar{t}_{ik})|| \leq r \text{ if } \mathcal{S}_i(k) \cap \mathcal{S}_j(q) \not\succ \tau_S \tag{31}$$

Inequalities (30) and (31) are consequences of the assumption that $j \in \mathcal{N}_i$. These inequalities will translate into constraints on the state of $\mathbb{S}$.

4.1.1 Definition of $\mathbb{S}$

We will take as the state space of $\mathbb{S}$, the space $\mathcal{X}$ of all lists $\{y_1, y_2, \dots y_n, w_1, w_2, \dots, w_n\}$ satisfying

$$\left. \begin{array}{l} y_i, w_i \in \mathbb{R}^2, \\ \\ ||y_i - y_j|| \leq r \end{array} \right\} j \in \mathcal{N}_i, \quad i \in \{1, 2, \dots, n\} \tag{32}$$

In the sequel we often write y for $\{y_1, y_2, \dots y_n\}$ and w for $\{w_1, w_2, \dots w_n\}$. We sometimes refer to $\{y_i, w_i\}$ as the state of "node" i. For $i \in \{1, 2, \dots, n\}$ let P_i^{-1} be a left inverse of P_i and let $\mathcal{P}_i = \mathcal{P} \cap \text{image } P_i$. We now define $\mathbb{S}$ to be a time-varying system with state $\{y, w\}$; for each $i \in \{1, 2, \dots, n\}$, the state of node i evolves on $\mathcal{P}$ according to update equations defined for $p \in \mathcal{P}_i$ by

$$y_i(p+1) = y_i(p) + u_{m_i}(v_{ii_1}(p) - y_i(p), \dots, v_{ii_{m_i}}(p) - y_i(p)) \tag{33}$$

$$w_i(p+1) = y_i(p) \tag{34}$$

where

$$v_{ij}(p) = \left. \begin{cases} y_j(p) & \text{if } \mathcal{S}_i(P_i^{-1}(p)) \cap \mathcal{S}_j(\lceil t_p \rceil_j) \succ \tau_S \\ w_j(p) & \text{otherwise} \end{cases} \right\}, \qquad j \in \mathcal{N}_i \tag{35}$$

and by

$$y_i(p+1) = y_i(p) \tag{36}$$

$$w_i(p+1) = w_i(p) \tag{37}$$

for $p \notin \mathcal{P}_i$. We require y_i satisfies the *neighbor constraints*

$$||y_i(p) - w_j(p)|| \leq r \text{ if } \mathcal{S}_i(P_i^{-1}(p)) \cap \mathcal{S}_j(\lceil t_p \rceil_j) \not\succ \tau_S, \quad p \in \mathcal{P}_i \quad j \in \mathcal{N}_i \tag{38}$$

Note that these constraint requirements together with the definition of $\mathcal{X}$ and v_{ij} insure that $||v_{ij} - y_i(p)|| \leq r$ whenever $p \in \mathcal{P}_i$. This in turn is necessary for (33) to make sense because the domain of u_{m_i} is $\mathbb{D}^{m_i}$.

The preceding defines $\mathbb{S}$ to be a synchronous discrete-time dynamical system with state constraints given by (38). The definition depends on the $\mathcal{N}_i$ as well as the n event time sequences $\{\bar{t}_{ik}; k \geq 1\}$. We've assumed that the $\mathcal{N}_i$ are non-empty; in addition, $\mathcal{N}_i \subset \{1, 2, \ldots, i-1, i+1, \ldots, n\}$. As a consequence of Proposition 2 and the assumption that neighbors stop changing, the $\mathcal{N}_i$ all have the following *symmetry property:* If $j \in \mathcal{N}_i$ then $i \in \mathcal{N}_j$. Because of the symmetry property we can associate with the $\mathcal{N}_i$ a simple graph $\mathbb{G}$ with vertex set $\{1, 2, \ldots, n\}$ and edge set defined in such a way that (i, j) is in the edge set just in case $i \in \mathcal{N}_j$ and $j \in \mathcal{N}_i$. Note that this is precisely the same as the simple graph mentioned just before Theorem 1. As for event times, recall that each event time sequence is strictly monotone increasing and that together they all satisfy Lemma 1, (2), and (3). In defining $\mathbb{S}$, these are the only properties of the $\mathcal{N}_i$ and the event times which are assumed.

4.1.2 Validation of $\mathbb{S}$

We claim that $\mathbb{S}$ provides a synchronous model of the asynchronous agent system describe by (27) - (31). The first step in justifying this claim is to define

$$\left.\begin{array}{rcl} y_i(p) & = & x_i(\bar{t}_{ik}) \\ w_i(p) & = & x_i(\bar{t}_{i(k-1)}) \end{array}\right\}, \quad P_i(k-1) < p \leq P_i(k), \;\; k \in P_i^{-1}(\mathcal{P}) \tag{39}$$

for $i \in \{1, 2, \ldots, n\}$. Note that y_i has been defined so that it is constant between agent i's event times and agrees with x_i whenever p is such that t_p is within one of agent i's sensing periods.

To justify the claim that $\mathbb{S}$ models (27) - (31), we need to prove that with the $y_i(p)$ and $w_i(p)\}$ defined by (39), $\{y(p), w(p)\} \in \mathcal{X}, \; p \in \mathcal{P}$, and (33) - (38) are satisfied. In view of (30) and the definition of the $y_i(p)$ in (2), it is clear that for $i \in \{1, 2, \ldots, n\}$, $||y_i(p) - y_j(p)|| \leq r, j \in \mathcal{N}_i, \; p \in \mathcal{P}$. Therefore $\{y(p), w(p)\} \in \mathcal{X}, \; p \in \mathcal{P}$. It remains to be shown that (33) - (38) are satisfied. To accomplish this, fix $p \in \mathcal{P}$ and suppose that k is that value for which $P_i(k) \leq p < P_i(k+1)$. Set $p_1 = P_i(k)$ and $p_2 = P_i(k+1)$. By definition,

$$y_i(p_1) = x_i(\bar{t}_{ik}) \tag{40}$$
$$w_i(p_1) = x_i(\bar{t}_{i(k-1)}) \tag{41}$$
$$y_i(p_2) = x_i(\bar{t}_{i(k+1)}) \tag{42}$$
$$w_i(p_2) = x_i(\bar{t}_{ik}) \tag{43}$$
$$y_i(s) = y_i(p_2), \;\; p_1 < s \leq p_2 \tag{44}$$
$$w_i(s) = w_i(p_2), \;\; p_1 < s \leq p_2 \tag{45}$$

Suppose first that $p \notin \mathcal{P}_i$, or equivalently that $p_1 < p < p_2$. Then $p_1 < p+1 \leq p_2$, so $y_i(p+1) = y_i(p_2)$ and $w_i(p+1) = w_i(p_2)$ because of (44) and (45) respectively. But $y_i(p) = y_i(p_2)$ and $w_i(p) = w_i(p_2)$ also because of (44) and (45) respectively. It follows that (36) and (37) are true.

Now suppose that $p \in \mathcal{P}_i$, or equivalently that $p = p_1$. Then $p_1 < p+1 \leq p_2$ so $y_i(p+1) = y_i(p_2)$ and $w_i(p+1) = w_i(p_2)$ because of (44) and (45) respectively. It follows from (42) and (43) that

$$y_i(p+1) = x_i(\bar{t}_{i(k+1)}) \tag{46}$$

and

$$w_i(p+1) = x_i(\bar{t}_{ik}) \tag{47}$$

But

$$x_i(\bar{t}_{ik}) = y_i(p) \tag{48}$$

because of (40) so (34) is true.

Fix $j \in \mathcal{N}_i$ and set $q = \lceil t_p \rceil_j$. To justify (38) and (33) we will need to express $x_j(\bar{t}_{iq})$, $x_j(\bar{t}_{i(q-1)})$ and k in terms of y_j, w_j and p respectively. Note first that $t_p = \bar{t}_{ik}$ because $p = p_1$. Thus

$$q = \lceil \bar{t}_{ik} \rceil_j \tag{49}$$

so $\bar{t}_{j(q-1)} < \bar{t}_{ik} \leq \bar{t}_{jq}$. This means that $P_j(q-1) < P_i(k) \leq P_j(q)$ and thus that $P_j(q-1) < p \leq P_j(q)$. But by definition $y_j(s) = x_j(\bar{t}_{jq})$ and $w_j(s) = x_j(\bar{t}_{j(q-1)})$ for $P_j(q-1) < s \leq P_j(q)$. Therefore

$$x_j(\bar{t}_{jq}) = y_j(p) \tag{50}$$

$$x_j(\bar{t}_{j(q-1)}) = w_j(p) \tag{51}$$

Finally note that

$$k = P_i^{-1}(p) \tag{52}$$

because $P_i(k) = p_1 = p$. It is now clear from (40), (51) and (52), that the inequality in (31) translates into neighbor constraint (38).

In addition, examination of (48) to (52) together with the definitions of $x_{ij}(\bar{t}_{ik})$ and $v_{ij}(p)$ in (28) and (35) respectively, reveals that

$$x_{ij}(\bar{t}_{ik}) = v_{ij}(p) \tag{53}$$

From this and (48) it follows that the expression for $x_i(\bar{t}_{i(k+1)})$ in (27) can be written as

$$x_i(\bar{t}_{i(k+1)}) = y_i(p) + u_{m_i}(v_{ii_1}(p) - y_i(p), \ldots,\ v_{ii_{m_i}}(p) - y_i(p))$$

This and (46) thus finally justify (33).

By a *trajectory* of $\mathbb{S}$ is meant a sequence of states $\{\{y(p), w(p)\} : p \in \mathcal{P}\}$ which satisfy (33) - (37) as well as the neighbor constraints (38). The preceding proves that the family of such trajectories is non-empty and contains the trajectory which represents actual agent system under consideration. It turns out that the trajectory representing the actual agent system has an additional property which we will exploit later.

Lemma 4 *For $i \in \{1, 2, \ldots, n\}$, let $y_i(p)$ and $w_i(p)$ be defined by (39). Let $i \in \{1, 2, \ldots, n\}$ and $s \in \mathcal{S}_i$ be fixed. Suppose that for some $j \in \{1, 2, \ldots, n\}$, and $p \in \mathcal{P}_i$*

$$||y_i(p+1) - y_j(p)|| \leq r \tag{54}$$

$$||w_i(p+1) - y_j(p)|| \leq r \tag{55}$$

Then $j \in \mathcal{N}_i$

Proof of Lemma 4: Since $p \in \mathcal{P}_i$ and P_i is strictly monotone, there is a unique integer k for which $p = P_i(k)$. Let $q = \lceil \bar{t}_{ik} \rceil$. As was noted previously in the development leading to (46) to

(50) $y_i(p+1) = x_i(\bar{t}_{i(k+1)})$, $w_i(p) = x_i(\bar{t}_{ik})$ and $y_j(p) = x_j(\bar{t}_{jq})$. Thus (54) and (55) translate into $||x_i(\bar{t}_{i(k+1)}) - x_j(\bar{t}_{jq})|| \leq r$ and $||x_i(\bar{t}_{ik}) - x_j(\bar{t}_{jq})|| \leq r$ respectively. Moreover Lemma 1 states that $\mathcal{S}_i(k)$ must strongly overlap either $\mathcal{S}_j(q)$ or $\mathcal{S}_j(q-1)$. If the former is true, then condition A of Proposition 1 is satisfied so $j \in \mathcal{N}_i$. Suppose next that $\mathcal{S}_i(k)$ does not strongly overlap $\mathcal{S}_j(q)$. Then $\bar{t}_{i(k+1)} \in \{q, q+1\}$ because of (4) and condition 3 in Lemma 2. If $\bar{t}_{i(k+1)} = q$, then $\mathcal{S}_i(k+1) \cap \mathcal{S}_j(q) \geq \tau_S$ because of condition 1 in Lemma 2. Thus condition A of Lemma 1 is satisfied when $k+1$ is substituted for k so in this case $j \in \mathcal{N}_i$. Suppose $\bar{t}_{i(k+1)} = q+1$. In view of Lemma 2, $\mathcal{S}_i(k)$ and $\mathcal{S}_i(k+1)$ are the only sensing periods of agent i which can strongly overlap $\mathcal{S}_j(q)$. Since $\mathcal{S}_j(q)$ must be strongly overlapped by at least one of agent i's sensing periods, it must be true that $\mathcal{S}_i(k+1) \cap \mathcal{S}_j(q) \geq \tau_S$. Thus condition B of Lemma 1 is satisfied with $k+1$ and $q+1$ substituted for k and q respectively. Therefore $j \in \mathcal{N}_i$. ∎

Conditions (54) and(55) do not necessarily imply that $j \in \mathcal{N}_i$ for every trajectory of $\mathbb{S}$. The claim of Lemma 4 is that the implication does indeed hold if the trajectory in question is the one which models the actual agent system.

4.2 Properties of $\mathbb{S}$

In Section 4.1 we defined $\mathbb{S}$ and proved that it faithfully models the actual agent system. In this section we derive several important properties of $\mathbb{S}$.

4.2.1 Local Convex Hulls

In the sequel we denote the convex hull of a given set of points $x_1, x_2, \ldots, x_q$ in $\mathbb{R}^2$ by $\langle x_1, x_2, \ldots, x_q \rangle$. We write $\mathcal{H}_i(p)$ for the *ith local convex hull*

$$\mathcal{H}_i(p) = \langle y_i(p), y_{i_1}(p), \ldots, y_{i_{m_i}}(p), w_i(p), w_{i_1}(p), \ldots, w_{i_{m_i}}(p) \rangle$$

where $\{i_1, i_2, \ldots, i_{m_i}\} = \mathcal{N}_i$. We also write $\mathcal{H}(p)$ for the {global} convex hull

$$\mathcal{H}(p) = \langle y_1(p), y_2(p), \ldots, y_n(p), w_1(p), w_2(p), \ldots, w_n(p) \rangle,$$

and $\mathcal{K}(p)$ for the set of corners of $\mathcal{H}(p)$. Clearly

$$\mathcal{H}_i(p) \subset \mathcal{H}(p), \qquad i \in \{1, 2, \ldots, n\}, \quad p \in \mathcal{P} \tag{56}$$

This fact plays a role in the proof of the following lemma which established a fundamental property of $\mathbb{S}$:

Lemma 5

$$\mathcal{H}(p+1) \subset \mathcal{H}(p), \quad p \in \mathcal{P} \tag{57}$$

Proof of Lemma 5: Fix $i \in \{1, 2, \ldots, n\}$ and note that (33) and the control law requirement that $u_m(z_1, z_2, \ldots, z_m) \in \langle 0, z_1, \ldots, z_m \rangle$, $z_i \in \mathbb{D}$, imply that $y_i(p+1) \in \mathcal{H}_i(p) \quad p \in \mathcal{P}_i$; thus $y_i(p+1) \in \mathcal{H}(p), \quad p \in \mathcal{P}_i$ because of (56). Moreover $y_i(p+1)$ is also in $\mathcal{H}(p)$ for $p \notin \mathcal{P}_i$ because of (36). Therefore $y_i(p+1) \in \mathcal{H}(p), \ \forall p \in \mathcal{P}$. Similarly $w_i(p+1) \in \mathcal{H}(p), \ p \in \mathcal{P}$ because of (34) and (37). Thus $\{y_i(p+1), w_i(p+1)\} \subset \mathcal{H}(p), \ p \in \mathcal{P}$. Since this holds for all $i \in \{1, 2, \ldots, n\}$, (57) is true. ∎

4.2.2 Stationary Nodes

Let us agree to say that node i is *stationary* at time $p \in \mathcal{P}_i$ if

$$y_i(p) = v_{ii_1}(p) = \cdots = v_{ii_{m_i}}(p)$$

The terminology is prompted by the fact that if node i is stationary at p, then $y_i(p+1) = y_i(p)$; this can be seen from (33) and the control law requirements imposed on u_{m_i}. In addition, the requirement that $u_m(z_1, z_2, \ldots, z_m)$ not be a corner of $\langle 0, z_1, \ldots, z_m \rangle$ unless $z_1 = z_2 = \cdots z_m = 0$, implies that if $y_i(p+1)$ is a corner of $\langle y_i(p), v_{ii_1}(p), \ldots, v_{ii_{m_i}}(p) \rangle$, then node i must be stationary at p. The following lemma implies that this is also true if $y_i(p+1)$ is a corner of $\mathcal{H}(p)$.

Lemma 6 *Fix $i \in \{1, 2, \ldots, n\}$ and $\bar{p} \in \mathcal{P}_i$. If $y_i(\bar{p}+1) \in \mathcal{K}(\widehat{p})$ for some $\widehat{p} \leq \bar{p}$, then node i must be stationary at each $p \in \mathcal{P}_i \cap \{p : \widehat{p} \leq p \leq \bar{p}\}$ and*

$$y_i(p) = y_i(\bar{p}+1), \tag{58}$$

for all such p.

Proof of Lemma 6 Let $p_1, p_2, \ldots, p_m$ denote the elements of the set $\mathcal{P}_i \cap \{p : \widehat{p} \leq p \leq \bar{p}\}$, ordered so that $p_1 < p_2 < \cdots < p_m = \bar{p}$. To prove the lemma it is sufficient to show that the statements

i. Node i is stationary at $p_k, p_{k+1}, \ldots, p_m$.

ii. $y_i(p_k) = y_i(p_{k+1}) = \cdots = y_i(p_m) = y_i(\bar{p}+1)$

hold for $k \in \{1, 2, \ldots, m\}$.

Let $\bar{\mathcal{H}}(p_s) = \langle y_i(p_s), v_{ii_1}(p_s), \ldots, v_{ii_{m_i}}(p_s) \rangle$, $s \in \{1, 2, \ldots, m\}$. Note that u_m must satisfy the control law requirement $u_m(z_1, z_2, \ldots, z_m) \in \langle 0, z_1, \ldots, z_m \rangle$. In view of (33), it must therefore be true that

$$y_i(p_s+1) \in \bar{\mathcal{H}}(p_s), \quad s \in \{1, 2, \ldots, m\} \tag{59}$$

Note next that the definition of v_{ij} in (35) implies that $v_{ij}(p_s) \in \{y_j(p_s), w_j(p_s)\}$, $s \in \{1, 2, \ldots, m\}$. Therefore $\bar{\mathcal{H}}(p_s) \subset \mathcal{H}_i(p_s)$. But $\mathcal{H}_i(p_s) \subset \mathcal{H}(p_s)$; moreover $\mathcal{H}(p_s) \subset \mathcal{H}(\widehat{p})$ because of Lemma 5. Thus $\bar{\mathcal{H}}(p_s) \subset \mathcal{H}(\widehat{p})$. This implies that

$$\bar{\mathcal{H}}(p_s) \cap \mathcal{K}(\widehat{p}) \subset \bar{\mathcal{K}}(p_s), \quad s \in \{1, 2, \ldots, m\} \tag{60}$$

where $\bar{\mathcal{K}}(p_s)$ is the corner set of $\bar{\mathcal{H}}(p_s)$.

Recall that $p_m = \bar{p}$. By assumption, $y_i(\bar{p}+1) \in \mathcal{K}(\widehat{p})$. These facts and (59) imply that $y_i(p_m+1) \in \bar{\mathcal{H}}(p_m) \cap \mathcal{K}(\widehat{p})$. Thus $y_i(p_m+1) \in \bar{\mathcal{K}}(p_m)$ because of (60). Therefore node i is stationary at p_m and because of this $y_i(p_m+1) = y_i(p_m)$. Thus statements i. and ii. above are true for $k = m$. If $m = 1$ the proof is complete.

Suppose next that $m > 1$ and that statements i. and ii. hold for all $k \in \{q+1, \ldots, m\}$ where q is some integer satisfying $1 < q+1 \leq m$. In view of (36), $y_i(p) = y_i(p_{q+1})$ for $p_q < p \leq p_{q+1}$. Therefore

$$y_i(p_q+1) = y_i(p_{q+1}) \tag{61}$$

By hypothesis, ii. holds for $k = q+1$; thus $y_i(p_q+1) = y_i(\bar{p}+1)$. Therefore $y_i(p_q+1) \in \mathcal{K}(\widehat{p})$. But $y_i(p_q+1) \in \bar{\mathcal{H}}(p_q)$ because of (59). Therefore $y_i(p_q+1) \in \bar{\mathcal{H}}(p_q) \cap \mathcal{K}(\widehat{p})$. From this and (60) it follows that $y_i(p_q+1) \in \bar{\mathcal{K}}(p_q)$. Therefore node i is stationary at p_q and because of this $y_i(p_q+1) = y_i(p_q)$. Hence $y_i(p_q) = y_i(p_{q+1})$ because of (61). Thus statements i. and ii. above are true for $k = \{q, q+1, \ldots, m\}$. By induction, statements i. and ii. must hold for all $k \in \{1, 2, \ldots, m\}$. ∎

4.2.3 Equilibrium States

By an *equilibrium state* of $\mathbb{S}$ we mean a state which does not change under the action of (33) - (37) under any conditions for every value of $p \in \mathcal{P}$. It is easy to see that equilibrium states are precisely those states $\{y, w\} \in \mathcal{X}$ for which

$$y_i = y_{ii_1} = \cdots = y_{ii_{m_i}} = w_i = w_{ii_1} \cdots = w_{ii_{m_i}}, \quad \forall i \in \{1, 2, \ldots, n\}$$

Note that each equilibrium state is invariant set under the action of (33) - (37) under any and all possible conditions. It is clear that if $\mathbb{S}$ is in an equilibrium state at p, then each node of $\mathbb{S}$ is stationary at p. It is also not difficult to see that if each node of $\mathbb{S}$ is stationary at p, then $\mathbb{S}$ is at an equilibrium state at time $p+1$.

4.2.4 Locally Rendezvoused Nodes

In the sequel we will say node $i \in \{1, 2, \ldots, n\}$ has *locally rendezvoused* at time p if $\mathcal{H}_i(p)$ is a single point; i.e., if $y_i(p) = y_{i_1}(p) = \cdots = y_{i_{m_i}}(p) = w_i(p) = w_{i_1}(p) = \cdots = w_{i_{m_i}}(p)$. Note that if a node has locally rendezvoused at p it must be stationary at p. The following proposition provides a criterion for a node of $\mathbb{S}$ to be locally rendezvoused.

Proposition 4 *Let $p_1 < p_2 < p_3 < p_4$ be four consecutive values of p in $\mathcal{P}_i$. If $y_i(p_4+1) \in \mathcal{K}(p_1)$, then node i is locally rendezvoused at $p = p_3$.*

The proof of Proposition 4 depends on the following lemmas.

Lemma 7 *Let p_1 and p_2 be two consecutive values of p in $\mathcal{P}_i$. Suppose for some $i \in \{1, 2, \ldots, n\}$, that $y_i(p_2+1) \in \mathcal{K}(p_1)$. Then*

$$y_i(p_1) = y_j(p_1), \quad j \in \mathcal{N}_i \tag{62}$$

Proof of Lemma 7: By hypothesis, $y_i(p_2+1) \in \mathcal{K}(p_1)$. Therefore by Lemma 6, $y_i(p_1) = y_i(p_2)$ and node i is stationary at both p_1 and p_2. Because node i is stationary at p_2, $y_i(p_2) = v_{ij}(p_2),\ j \in \mathcal{N}_i$. Therefore

$$y_i(p_1) = v_{ij}(p_2), \ j \in \mathcal{N}_i \tag{63}$$

The justify (62) it is therefore enough to show that

$$v_{ij}(p_2) = y_j(p_1), \ j \in \mathcal{N}_i \tag{64}$$

For this fix $j \in \mathcal{N}_i$ and define $k = P_i^{-1}(p_1)$ and $q = \lceil \bar{t}_{ik} \rceil_j$. Since $p_1 = P_i(k)$ and $\bar{t}_{j(q-1)} < \bar{t}_{ik} \leq \bar{t}_{jq}$,

$$P_j(q-1) < p_1 \leq P_j(q) \tag{65}$$

Let $\bar{q} = \lceil \bar{t}_{i(k+1)} \rceil_j$. Since $p_2 = P_i(k+1)$ and $\bar{t}_{j(\bar{q}-1)} < \bar{t}_{i(k+1)} \leq \bar{t}_{j\bar{q}}$,

$$P_j(\bar{q}-1) < p_2 \leq P_j(\bar{q}) \tag{66}$$

By Lemma 2, $\bar{q} \in \{q, q+1, q+2\}$. We claim that no matter which value $\bar{q}$ takes,

$$v_{ij}(p_2) \in \{y_j(P_j(q)),\ y_j(P_j(q+1)),\ y_j(P_j(q+2))\} \tag{67}$$

To justify this claim, consider first the case when $\bar{q} = q$. Then $\mathcal{S}_i(k+1) \cap \mathcal{S}_j(q) \succ \tau_S$ because of Lemma 2. In general $\bar{q} = \lceil t_{p_2} \rceil_j$ because $t_{p_2} = \bar{t}_{i(k+1)}$. Thus in this case $q = \lceil t_{p_2} \rceil_j$. In addition $k+1 = P_i^{-1}(p_2)$. Therefore $\mathcal{S}_i(P_i^{-1}(p_2)) \cap \mathcal{S}_j(\lceil t_{p_2} \rceil_j) \succ \tau_S$. From this and (35) it follows that $v_{ij}(p_2) = y_j(p_2)$. In view of (36), $y_j(p) = y_j(P_j(q))$ for all values of p in the range $P_j(q-1) < p \leq P_j(q)$. But $P_j(q-1) < p_2 \leq P_j(q)$ because of (66). Therefore $y_j(p_2) = y_j(P_j(q))$. Thus $v_{ij}(p_2) = y_j(P_j(q))$ which proves that (67) holds in this case.

Now suppose that $\bar{q} = \{q+1, q+2\}$. In this case $v_{ij}(p_2)$ equals either $y_j(p_2)$ or $w_j(p_2)$ because of (35). In view of (36), $y_j(p) = y_j(P_j(\bar{q}))$ for $P_j(\bar{q}-1) < p \leq P_j(\bar{q})$. From this and (66) it follows that $y_j(p_2) = y_j(P_j(\bar{q}))$. Thus if $v_{ij}(p_2) = y_j(p_2)$ then $v_{ij}(p_2) = y_j(P_j(\bar{q}))$. Since $\bar{q} \in \{q+1, q+2\}$, (67) must hold in this situation. To prove that (67) also holds in the alternative situation, when $v_{ij}(p_2) = w_j(p_2)$, we exploit the relation $w_j(P_j(\bar{q}-1)+1) = y_j(P_j(\bar{q}-1))$ which is valid because of (34). In view of (37), $w_j(p)$ is constant for p in the range $P_j(\bar{q}-1) < p \leq P_j(\bar{q})$. But p_2 is in this range because of (66); clearly $P_j(\bar{q}-1)+1$ is as well. Therefore $w_j(p_2) = w_j(P_j(\bar{q}-1)+1)$. It follows that $w_j(p_2) = y_j(P_j(\bar{q}-1))$. Thus if $v_{ij}(p_2) = w_j(p_2)$ then $v_{ij}(p_2) = y_j(P_j(\bar{q}-1))$. Since $\bar{q} \in \{q+1, q+2\}$, (67) must hold in this situation too. Thus (67) holds under all conditions.

It will now be shown that

$$v_{ij}(p_2) = y_j(P_j(q)) \tag{68}$$

Consider first the situation when $v_{ij}(p_2) = y_j(P_j(s))$ where s is fixed at either value in $\{q+1, q+2\}$. Since node i is stationary at p_2, $v_{ij}(p_2) = y_i(p_2+1)$. Thus $y_j(P_j(s)) = y_i(p_2+1)$. By hypothesis, $y_i(p_2+1) \in \mathcal{K}(p_1)$. Thus $y_j(P_j(s)) \in \mathcal{K}(p_1)$. Moreover $p_1 \leq P_j(q)$ because of (65). Thus by Lemma 6, $y_j(P_j(s)) = y_j(P_j(q))$. Therefore (68) holds when $v_{ij}(p_2) = y_j(P_j(s))$ for $s \in \{q+1, q+2\}$. In view of (67), the only other possibility is $v_{ij}(p_2) = y_j(P_j(q))$. Therefore (68) is true under all conditions.

It remains to be shown that (64) holds. In view of (36), $y_j(p) = y_j(P_j(q))$ for p in the range $P_j(q-1) < p \leq P_j(q)$. But (65) shows that p_1 is in this range so $y_j(p_1) = y_j(P_j(q))$. From this and (68) it follows that (64) holds. ■

Lemma 8 *For any integers $i \in \{1, 2, \ldots, n\}$ and $k \geq 1$*

$$P_i(k+1) - P_i(k) \leq 2(n-1) \tag{69}$$

Moreover for any integer $j \in \{1, 2, \ldots, n\}$ which is not equal to i, there are at most two successive positive integers $s, s+1$ such that

$$P_i(k) \leq P_j(s) < P_j(s+1) \leq P_i(k+1) \tag{70}$$

Proof of Lemma 8: Fix $i, j \in \{1, 2, \dots, n\}$ and $k > 0$. Let s and p be positive integers such that $\bar{t}_{ik} \leq \bar{t}_{js} < \bar{t}_{j(s+p)} \leq \bar{t}_{i(k+1)}$. These inequalities imply that $\bar{t}_{j(s+p)} - \bar{t}_{js} < \bar{t}_{i(k+1)} - \bar{t}_{ik}$. But $p\tau_D \leq \bar{t}_{j(s+p)} - \bar{t}_{js}$ because of (2) and $\bar{t}_{i(k+1)} - \bar{t}_{ik} < 2\tau_D$ because of (3). Therefore $p\tau_D < 2\tau_D$ so $p = 1$. Thus there are at most two successive event times $\bar{t}_{js}$ and $\bar{t}_{j(s+1)}$ for which $\bar{t}_{ik} \leq \bar{t}_{js} < \bar{t}_{j(s+1)} \leq \bar{t}_{i(k+1)}$. Moreover, since $\{j : j \in \{1, 2, \dots, n\},\ j \neq i\}$ contains $n - 1$ integers, it therefore follows that number of distinct event times in the set $\{\bar{t}_{js} : j \in \{1, 2, \dots, n\},\ j \neq i,\ s \geq 1\}$ which satisfy $\bar{t}_{ik} \leq \bar{t}_{js} \leq \bar{t}_{i(k+1)}$ does not exceed $2(n - 1)$. But $P_i(\cdot)$ and $P_j(\cdot)$ are strictly monotone increasing and $\bar{t}_{iq} = t_{P_i(q)}$, $\bar{t}_{jq} = t_{P_j(q)}$ for all $q \geq 1$. Therefore (69) is true and there are at most two successive integers $s, s + 1$ for which (70) holds ■

Proof of Proposition 4: By hypothesis, $y_i(p_4 + 1) \in \mathcal{K}(p_1)$ and $p_1 < p_2 < p_3 < p_4$. Therefore by Lemma 6,

$$y_i(p_2) = y_i(p_3) = y_i(p_4) = y_i(p_4 + 1) \tag{71}$$

and node i is stationary at p_3 and p_4. In view of (34), $w_i(p_2 + 1) = y_i(p_2)$. But $w_i(p) = w_i(p_3)$ for $p_2 < p \leq p_3$ because of (37), so $w_i(p_2 + 1) = w_i(p_3)$. Therefore $y_i(p_2) = w_i(p_3)$. From this and (71) it follows that

$$y_i(p_3) = w_i(p_3) \tag{72}$$

By hypothesis $y_i(p_4 + 1) \in \mathcal{K}(p_1)$. In addition, $y_i(p_4 + 1) \in \mathcal{H}(p_3)$ because of (71). Thus $y_i(p_4 + 1) \in \mathcal{K}(p_1) \cap \mathcal{H}(p_3)$. In view of Lemma 5, $\mathcal{H}(p_3) \subset \mathcal{H}(p_1)$. Thus $\mathcal{K}(p_1) \cap \mathcal{H}(p_3) \subset \mathcal{K}(p_3)$. Therefore $y_i(p_4 + 1) \in \mathcal{K}(p_3)$. Hence by Lemma 7,

$$y_i(p_3) = y_j(p_3), \quad j \in \mathcal{N}_i \tag{73}$$

If view of (72) and (73), node i will be rendezvoused at p_3 provided

$$y_j(p_3) = w_j(p_3), \ \ j \in \mathcal{N}_i \tag{74}$$

It will now be shown that this is true.

Fix $j \in \mathcal{N}_i$ and let $q = \lceil \bar{t}_{ik} \rceil_j$ where $k = P_i^{-1}(p_3)$. Equivalently, q is the unique integer for which $P_j(q - 1) < p_3 \leq P_j(q)$. In view of (36)and (37), $y_j(p)$ and $w_j(p)$ are constant for p in the range $P_j(q - 1) < p \leq P_j(q)$. Since both p_3 and $P_j(q - 1) + 1$ are in this range,

$$y_j(p_3) = y_j(P_j(q - 1) + 1) \qquad \text{and} \qquad w_j(p_3) = w_j(P_j(q - 1) + 1) \tag{75}$$

Note next that $y_i(p_4 + 1) = y_i(p_4)$ because node i is stationary at p_4. From this and (71) and (73) it follows that $y_i(p_4 + 1) = y_j(p_3)$. Thus $y_i(p_4 + 1) = y_j(P_j(q - 1) + 1)$. Since $y_i(p_4 + 1) \in \mathcal{K}(p_1)$ it must be true that

$$y_j(P_j(q - 1) + 1) \in \mathcal{K}(p_1) \tag{76}$$

In view of Lemma 8, there can be at most two consecutive integers in $\mathcal{P}_j$ which are in the set $\{p : P_j(q-1) \leq p \leq P_j(q)\}$. Since p_3 is one such integer, it must be true that p_1 is not in the set. Therefore $p_1 < P_j(q - 1)$. From this, (76) and Lemma 6 it follows that $y_j(P_j(q - 1) + 1) = y_j(P_j(q - 1))$. But $w_j(P_j(q - 1) + 1) = y_j(P_j(q - 1))$ because of (34), so $w_j(P_j(q - 1) + 1) = y_j(P_j(q - 1) + 1)$. From this and (75) if follows that $w_j(p_3) = y_j(p_3)$. Therefore (74) is true. ■

4.3 Error system

To analyze system behavior it is helpful to use a suitably defined error system $\bar{\mathbb{S}}$ derived from $\mathbb{S}$. Towards this end, for each $p \in \mathcal{P}$ let

$$\left.\begin{array}{rcl} \bar{y}_i(p) & = & y_i(p) - w_n(p) \\ \\ \bar{w}_i(p) & = & w_i(p) - w_n(p) \end{array}\right\} i \in \{1, 2, \dots, n\} \tag{77}$$

Note that

$$\bar{w}_n(p) = 0, \quad p \in \mathcal{P} \tag{78}$$

Using (33) - (37) we obtain the update equations for $\{\bar{y}_i, \bar{w}_i\}$ defined for $p \in \mathcal{P}_i$ by

$$\bar{y}_i(p+1) = \bar{y}_i(p) + u_{m_i}(\bar{v}_{ii_1}(p) - \bar{y}_i(p), \dots,\ \bar{v}_{ii_{m_i}}(p) - \bar{y}_i(p)) - \omega(p)\bar{y}_n(p) \tag{79}$$

$$\bar{w}_i(p+1) = \bar{y}_i(p) - \omega(p)\bar{y}_n(p) \tag{80}$$

where

$$\bar{v}_{ij}(p) = \left\{\begin{array}{ll} \bar{y}_j(p) & \text{if } \mathcal{S}_i(P_i^{-1}(p)) \cap \mathcal{S}_j(\lceil t_p \rceil_j) \succ \tau_S \\ \bar{w}_j(p) & \text{otherwise} \end{array}\right\}, \qquad j \in \mathcal{N}_i \tag{81}$$

and by

$$\bar{y}_i(p+1) = \bar{y}_i(p) - \omega(p)\bar{y}_n(p) \tag{82}$$

$$\bar{w}_i(p+1) = \bar{w}_i(p) - \omega(p)\bar{y}_n(p) \tag{83}$$

for $p \notin \mathcal{P}_i$. Here $\omega(p) = 1$ if $p \in \mathcal{P}_n$ and $\omega(p) = 0$ otherwise. In terms of error variables, the neighbor constraints given by (38) can be written as

$$||\bar{y}_i(p) - \bar{w}_j(p)|| \leq r \ \text{ if } \quad \mathcal{S}_i(P_i^{-1}(p)) \cap \mathcal{S}_j(\lceil t_p \rceil_j) \not\succ \tau_S, \ \ p \in \mathcal{P}_i \ \ j \in \mathcal{N}_i \tag{84}$$

In the sequel $\bar{\mathbb{S}}$ denotes the error system defined by (79) - (84). Note that the state of $\bar{\mathbb{S}}$, namely $\{\bar{y}_1(p), \dots, \bar{y}_n(p), \bar{w}_1(p), \dots, \bar{w}_{n-1}(p)\}$, takes values in the closed space $\bar{\mathcal{X}}$ of all lists $\{\bar{y}_1, \dots, \bar{y}_n, \bar{w}_1, \dots, \bar{w}_{n-1}\}$ satisfying

$$\left.\begin{array}{l} \bar{y}_i, \bar{w}_i \in \mathbb{R}^2, \\ \\ ||\bar{y}_i - \bar{y}_j|| \leq r \end{array}\right\} j \in \mathcal{N}_i, \quad i \in \{1, 2, \dots, n\} \tag{85}$$

It is possible to describe the preceding state update equations concisely as

$$\bar{x}(p+1) = f(p, \bar{x}(p)), \quad p \in \mathcal{P}$$

where $\bar{x}$ is the state $\{\bar{y}_1, \dots, \bar{y}_n, \bar{w}_1, \dots, \bar{w}_{n-1}\}$, $f(p, \cdot) : \bar{\mathcal{X}}(p) \to \bar{\mathcal{X}}$ is the next state map defined by (79) - (83), and $\bar{\mathcal{X}}(p)$ is the set of states in $\bar{\mathcal{X}}$ for which then neighbor constraints (84) hold at time p. It is important to recognize that even though there are infinitely many possible values of p, there are only finitely many distinct $\bar{\mathcal{X}}(p)$ and finitely many distinct $f(p, \cdot)$. Moreover, each $\bar{\mathcal{X}}(p)$ is closed because of (84) and each $f(p, \cdot)$ is continuous on its domain because each $u_m(\cdot)$ is. The following lemma summarizes these observations.

Lemma 9 *There exists a finite index set $\mathcal{Q}$, and a finite set of continuous functions $F_q : \mathcal{X}_q \rightarrow \bar{\mathcal{X}}$ with closed domains such that the following statement is true. For any $p \in \mathcal{P}$ there is a $q \in \mathcal{Q}$ such that $\bar{\mathcal{X}}(p) = \mathcal{X}_q$ and $F_q(\cdot) = f(p, \cdot)$.*

The implication of Lemma 9 is that if $\{\bar{x}(p) : p \in \mathcal{P}\}$ is a trajectory of $\bar{\mathbb{S}}$, then there are indices $q(p) \in \mathcal{Q},\ p \in \mathcal{P}$ such that

$$\bar{x}(p) = F_{q(p)} F_{q(p-1)} \cdots F_{q(\tau+1)}(\bar{x}(\tau)),\ \ p > \tau,\ \ p, \tau \in \mathcal{P} \tag{86}$$

Here $F_{q(p)} F_{q(p-1)} \cdots F_{q(\tau+1)}$ is a "composed function", where by the composition of functions F_s and F_q we mean the function $F_q F_s : \mathcal{X}_{qs} \rightarrow \bar{\mathcal{X}}$, whose domain $\mathcal{X}_{qs}$ is the inverse image of $\mathcal{X}_q$ under F_s, and whose action on $\bar{x}$ is $\bar{x} \longmapsto F_q(F_s(\bar{x}))$. Composition is an associative operation and because of this, the operation extends unambiguously to finite families of F_q. Note that any such composed function $F = F_{q_1} F_{q_2} \cdots F_{q_k}$ has a closed domain on which it is continuous.

Suppose that $\bar{p} > 0$ is fixed. If follows from the preceding that there are $q(p) \in \mathcal{Q}$ such that

$$\bar{x}(p + \bar{p}) = F_{q(p+\bar{p})} F_{q(p+\bar{p}-1)} \cdots F_{q(p+1)}(\bar{x}(p)),\ \ p \in \mathcal{P} \tag{87}$$

It is important to recognize that even though the composed function $F_{q(p+\bar{p})} F_{q(p+\bar{p}-1)} \cdots F_{q(p+1)}(\bar{x}(p))$ depends on p, there can be only a finite number of such composed functions. This is because the family of maps $\{F_q : q \in \mathcal{Q}\}$ is a finite set and because the composed functions in question are all compositions of exactly $\bar{p}$ maps in the family. The following proposition summarizes these observations.

Proposition 5 *Let $\bar{p} > 0$ be fixed. There exist a finite index set $\bar{\mathcal{Q}}$, a finite set of closed subsets $\bar{\mathcal{X}}_q \subset \bar{\mathcal{X}}$, and a finite set of continuous maps $D_q : \bar{\mathcal{X}}_q \rightarrow \bar{\mathcal{X}}$, $q \in \bar{\mathcal{Q}}$ with the following property. For each trajectory $\{\bar{x}(p) : p \in \mathcal{P}\}$ of $\bar{\mathbb{S}}$, and each $p \in \mathcal{P}$, there is a $q \in \bar{\mathcal{Q}}$ such that*

$$\bar{x}(\bar{p} + p) = D_q(\bar{x}(p)) \tag{88}$$

4.4 Global Rendezvous

It is natural to say that the n nodes of $\mathbb{S}$ have {globally} *rendezvoused* at time p if $\mathcal{H}(p)$ is a single point; i.e., if $y_1(p) = y_2(p) = \cdots = y_n(p) = w_1(p) = w_2(p) = \cdots = w_n(p)$. In view of the definitions of t_p and the y_i and w_i in (39) it is clear that the rendezvousing of all n nodes at time p implies the rendezvousing of all n agents at time t_p. It is also clear that the rendezvousing of all n nodes at time p implies that each node has locally rendezvoused at p. Under certain conditions the converse is also true.

Lemma 10 *Suppose $\mathbb{G}$ is a connected graph. Suppose in addition that $\{\{y(p), w(p)\} : p \in \mathcal{P}\}$ is the trajectory of $\mathbb{S}$ defined by (39). If for some $i \in \{1, 2, \ldots, n\}$ and $p \in \mathcal{P}_i$, node i is locally rendezvoused, then the n nodes of $\mathbb{S}$ have globally rendezvoused.*

Proof of Lemma 10: Suppose node i is locally rendezvoused at $p \in \mathcal{P}_i$. Then $y_i(p) = y_j(p)$ and $w_i(p) = y_j(p),\ \ j \in \mathcal{N}_i$. Moreover, since node i is locally rendezvoused at p it must be stationary at

p. Therefore $y_i(p+1) = y_i(p)$; in addition, $w_i(p+1)) = y_i(p)$ because of (34). Thus $y_i(p+1) = y_j(p)$ and $w_i(p+1) = y_j(p), \quad j \in \mathcal{N}_i$. Fix $j \in \mathcal{N}_i$ and $k \in \mathcal{N}_j$. Then $||y_j(p) - y_k(p)|| \leq r$ because of the definition of $\mathcal{X}$. Therefore $||y_i(p+1) - y_k(p)|| \leq r$ and $||w_i(p+1) - y_k(p)|| \leq r$. It follows from Lemma 4 that $k \in \mathcal{N}_i$. Since this holds for every $k \in \mathcal{N}_j$ it must be true that $\mathcal{N}_j \subset \mathcal{N}_i$. Since j is arbitrary, this must be true for all $j \in \mathcal{N}_i$. Since $\mathbb{G}$ is connected, this can happen only if $\mathbb{G}$ is complete. Thus $\mathcal{N}_i = \{1, 2, \ldots, n\}$ which means that $\mathcal{H}_i(p) = \mathcal{H}(p)$. By hypothesis $\mathcal{H}_i(p)$ is a single point. Therefore $\mathcal{H}_i(p)$ is also a single point so the n nodes of $\mathbb{S}$ have globally rendezvoused. ■

Establishing the preceding result requires one to be able to conclude that if for some $i, j \in \{1, 2, \ldots, n\}$ and some $p \in \mathcal{P}_i$, nodes i and j are in the same "position" in the sense that $y_i(p) = y_j(p) = w_i(p)$, then $\mathcal{N}_j \subset \mathcal{N}_i$. In words, what this is roughly saying is that if node j is in the same position as node i, then node j's "neighbors" must also be neighbors of node i. This *transitivity property* is not true in general but it is true if $y(p)$ and $w(p)$ are defined by (39) respectively. This is a consequence of the Lemma 4.

The following proposition shows that if $\mathcal{H}$ does not change for a sufficiently long period of time, then the n nodes have rendezvoused.

Proposition 6 *Suppose $\mathbb{G}$ is a connected graph. Suppose in addition that $\{y(p), w(p) : p \in \mathcal{P}\}$ is the trajectory of $\mathbb{S}$ defined by (39). Suppose that p_a and p_b are values in $\mathcal{P}$ for which $p_b - p_a \geq 8n$ and*

$$\text{dia}\{\mathcal{H}(p_a)\} = \text{dia}\{\mathcal{H}(p_b)\} \tag{89}$$

Then the n nodes of $\mathbb{S}$ have rendezvoused at $p = p_b$.

Proof of Proposition 6: Choose $i \in \{1, 2, \ldots, n\}$ so that for some $z \in \mathcal{H}(p_b)$, $||y_i(p_b) - z|| = \text{dia}\{\mathcal{H}(p_b)\}$. Then $y_i(p_b) \in \mathcal{K}(p_b)$. In view of Lemma 5, $\mathcal{H}(p_b) \subset \mathcal{H}(p_a)$. Therefore $y_i(p_b), z \in \mathcal{H}(p_a)$. Moreover $||y_i(p_b) - z|| = \text{dia}\{\mathcal{H}(p_a)\}$ because of (89) so

$$y_i(p_b) \in \mathcal{K}(p_a) \tag{90}$$

Let p_4 be the largest value of $p \in \mathcal{P}_i$ such that $p_4 < p_b$. Define $k = P_i^{-1}(p_4) - 3$ so $P_i(k+3) = p_4$. Then $p_4 < p_b \leq P_i(k+4)$. By (69),

$$p_b - p_4 \leq 2(n-1) \tag{91}$$

In view of (36), $y_i(p)$ is constant for p in the range $p_4 < p \leq P_i(k+4)$. Since both $p_4 + 1$ and p_b are in this range, $y_i(p_4 + 1) = y_i(p_b)$. Thus

$$y_i(p_4 + 1) \in \mathcal{K}(p_a) \tag{92}$$

Define $p_1 = P_i(k)$, $p_2 = P_i(k+1)$, and $p_3 = P_i(k+2)$. Clearly $p_1 < p_2 < p_3 < p_4$. Moreover $p_{j+1} - p_j \leq 2(n-1)$, $j \in \{1, 2, 3\}$ because of (69). From these inequalities and (91) it follows that $p_b - p_1 \leq 8(n-1)$. By hypothesis, $p_b - p_b \geq 8n$. Therefore $p_a < p_1$. In view of Lemma 5, $\mathcal{H}(p_4) \subset \mathcal{H}(p_1)$ and $\mathcal{H}(p_1) \subset \mathcal{H}(p_a)$. Therefore $\mathcal{H}(p_1) \cap \mathcal{K}(p_a) \subset \mathcal{K}(p_1)$. But $\mathcal{H}(p_4+1) \subset \mathcal{H}(p_1)$ because of Lemma 5;thus $y_i(p_4 + 1) \in \mathcal{H}(p_1)$. This and (92) imply that $y_i(p_4 + 1) \in \mathcal{H}(p_1) \cap \mathcal{K}(p_a)$. Therefore $y_i(p_4 + 1) \in \mathcal{K}(p_1)$. From this and Proposition 4 it follows that node i has locally rendezvoused at p_3. Therefore by Lemma 10, the n nodes of $\mathbb{S}$ are rendezvoused at p_3. ■

The following theorem is our main convergence result concerning $\mathbb{S}$. The main result of this paper, Theorem 1, is an immediate consequence.

Theorem 2 *Let $\{\{y(s), w(s)\} : p \in \mathcal{P}\}$ be the trajectory of $\mathbb{S}$ defined by (39). If $\mathbb{G}$ is a connected graph, then*

$$\lim_{s\to\infty} \mathrm{dia}\langle y_1(s), y_2(s), \ldots, y_n(s), w_1(s), w_2(s), \ldots, w_n(s)\rangle = 0 \tag{93}$$

Proof of Theorem 2: In the sequel we write $x(p)$ for $\{y_1(p), \ldots, y_n(p), w_1(p), \ldots, w_n(p)\}$ and $\bar{x}(p)$ for the error vector $\{\bar{y}_1(p), \ldots, \bar{y}_n(p), \bar{w}_1(p), \ldots, \bar{w}_{n-1}(p)\}$ defined by (77). Let $V : \mathcal{X} \to \mathrm{I\!R}$ denote the diameter function $x \longmapsto \mathrm{dia}\langle y_1, y_2, \ldots, y_n, w_1, w_2, \ldots, w_n\rangle$. Similarly, write $\bar{V} : \bar{\mathcal{X}} \to \mathrm{I\!R}$ denote the diameter function $\bar{x} \longmapsto \mathrm{dia}\langle \bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n, \bar{w}_1, \bar{w}_2, \ldots, w_{n-1}, 0\rangle$. Note that

$$V(x(p)) = \bar{V}(\bar{x}(p)) \tag{94}$$

Note in addition that because $0 \in \langle \bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n, \bar{w}_1, \bar{w}_2, \ldots, w_{n-1}, 0\rangle$, $\bar{V}$ is radially unbounded whereas V is not.

As a consequence of Lemma 5, $V(x(p))$ is a monotone non-increasing function of p. Clearly $V(x(p))$ is bounded below by 0. Moreover $V(x(p))$ is bounded above by $V(x(0))$ because $V(\cdot)$ is continuous. Therefore there must exist a finite limit

$$V^* = \lim_{p\to\infty} V(x(p))$$

We claim that $V^* = 0$. To prove this claim, suppose that is false. Then $V^* > 0$. This means that the trajectory $\{x(p) : p \in \mathcal{P}\}$ cannot contain any points in the set $\mathcal{E} = \{x : V(x) = 0\}$. To proceed, fix $\bar{s} > 8n$ and let $\Delta(x(p))$ denote the difference

$$\Delta(x(p)) = V(x(\bar{p} + p)) - V(x(p)) \tag{95}$$

Since $V(x(p))$ is monotone non-increasing, $\Delta(x(p)) \leq 0, \quad p \in \mathcal{P}$. In the light of Proposition 6 and the fact that $\mathcal{E}$ has no points in common with $\{x(p) : p \in \mathcal{P}\}$, one can conclude that $\Delta(x(p)) \neq 0, \quad p \in \mathcal{P}$. Therefore

$$\Delta(x(p)) < 0, \quad p \in \mathcal{P} \tag{96}$$

Define $\bar{\Delta}(\bar{x}(p))$ as

$$\bar{\Delta}(\bar{x}(p)) = \bar{V}(\bar{x}(\bar{p} + p)) - \bar{V}(\bar{x}(p)) \tag{97}$$

In view of (94)

$$\Delta(x(p)) = \bar{\Delta}(\bar{x}(p)) \tag{98}$$

Therefore

$$\bar{\Delta}(\bar{x}(p)) < 0, \quad p \in \mathcal{P} \tag{99}$$

According to Proposition 5, for each $p \in \mathcal{P}$ there is a continuous function D_q such that $\bar{x}(p + \bar{p}) = D_q(x(p))$. Let $\mathcal{W}_q$ denote the set of state pairs $(\bar{x}(p + \bar{p}), \bar{x}(p))$ along the given trajectory of $\bar{\mathbb{S}}$ for which this formula holds. It follows that

$$\{(x(s + \bar{s}), x(s)) : s \in \mathcal{S}\} = \bigcup_{q\in\mathcal{Q}} \mathcal{W}_q$$

and that each $\mathcal{W}_q$ is a closed set. We claim that each $\mathcal{W}_q$ is bounded as well. This is in fact so because of (94), because $\bar{V}$ is radially unbounded, and because $0 \leq V(x(p)) \leq V(x(0)) < \infty$.

For $(\widehat{x}, \bar{x}) \in \mathcal{W}_q$ define $\Delta_q : \mathcal{W}_q \rightarrow \mathbb{R}$ so that $(\widehat{x}, \bar{x}) \longmapsto \bar{V}(D_q(\widehat{x})) - V(\bar{x})$. Note that Δ_q is a continuous function on $\mathcal{W}_q$ whose value at each point $(\widehat{x}, \bar{x}) \in \mathcal{W}_q$ agrees with $\bar{\Delta}(\bar{x}(p))$ for some p. It follows from (99) that

$$\Delta_q(\widehat{x}, \bar{x}) < 0, \quad (\widehat{x}, \bar{x}) \in \mathcal{W}_q$$

Define

$$\mu_q = \sup_{(\widehat{x}, \bar{x}) \in \mathcal{W}_q} \Delta_q(\widehat{x}, \bar{x})$$

Since $\mathcal{W}_q$ is compact and Δ_q is negative and continuous on $\mathcal{W}_q$, it must be true that $\mu_q < 0$. Let

$$\mu = \max_{q \in \mathcal{Q}} \mu_i$$

Since $\mathcal{Q}$ is finite, $\mu < 0$. Clearly

$$\Delta_q(\widehat{x}, \bar{x}) \leq \mu \quad (\widehat{x}, \bar{x}) \in \mathcal{W}_q, \quad q \in \mathcal{Q} \tag{100}$$

Note that by construction, for each $p \in \mathcal{S}$ there must be a $q \in \mathcal{Q}$ such that $\bar{\Delta}(\bar{x}(p)) = \Delta_q(\bar{x}(p+\bar{p}), \bar{x}(p)))$ From this and (100) it follows that

$$\bar{\Delta}(\bar{x}(p)) \leq \mu, \quad p \in \mathcal{P}$$

Therefore

$$\Delta(x(p)) \leq \mu, \quad p \in \mathcal{P}$$

because of (98). Note that

$$V(x(p+\bar{p})) - V(x(p)) = \Delta(x(p)) \leq \mu, \quad p \in \mathcal{P}$$

Thus by summing,

$$V(x(p+k\bar{p})) \leq V(x(p)) + k\mu, \quad k \geq 1$$

Therefore, for k sufficiently large $V(x(p+k\bar{p}))$ must be negative because $\mu < 0$. But this is impossible because $V(\cdot)$ is positive semi-definite. Hence V^* cannot be positive. ■

5 Concluding Remarks

The analysis used in this paper exploits ideas which appear to have much in common with the embedding process discussed in Chapter 7 of [7] for analyzing "partially asynchronous iterative algorithms." This suggests that the tools developed in [7] may be helpful in understanding the asynchronous system considered in this paper.

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