Station Keeping in the Plane with Range-Only Measurements*

M. Cao A. S. Morse

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Abstract

Using concepts from switched adaptive control theory, provably correct solution is given to the three landmark station keeping problem in the plane in which range measurements are the only sensed signals upon which station keeping is to be based. The performance of the resulting system degrades gracefully in the face of measurement and miss-alignment errors, provided the measurement errors are not too large.

1 Introduction

"Station keeping" is a term from orbital mechanics which refers to the "practice of maintaining the orbital position of satellites in geostationary orbit" {Wikipedia}. In this paper we take station keeping to mean the practice of keeping a mobile autonomous agent in a prescribed position in the plane which is determined by prescribed distances from two or more landmarks. We refer to these landmarks as neighboring agents because we envision solutions to the station keeping problem as potential solutions to multi-agent formation maintenance problems. We are particularly interested in solutions to the station keeping problem in which the only signals available to the agent whose position is to be maintained, are noisy range measurements from its neighbors[1]. Work on this problem already exists [1, 2] and we anticipate more ideas from various research groups to emerge very soon. Our approach to station keeping is novel in that we treat station keeping as a problem in switched adaptive control. Related work on range-only source localization can be found in [3, 4].

In Section 2 we formulate the station keeping problem of interest. Error models appropriate to the solution to the problem are developed in Section 3. Some of the error equations developed have appeared previously in [1, 2, 5] and elsewhere. In Section 4 we present a switched adaptive control system which solves the three neighbor station keeping problem. Agent positioning in the absence of errors occurs exponentially fast while performance degrades gracefully in the face of measurement and miss-alignment errors, provided the measurement errors are not too large. In Section 5 we sketch the ideas upon which these claims are based. Finally in Section 6, we discuss possible approaches to an implementation issue which arises because the underlying parameter space appropriate to the problem is not typically convex.

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[1]We are indebted to B. D. O. Anderson for making us aware of this problem.

2 Formulation

Let $n > 1$ be an integer. The system of interest consists of $n + 1$ points in the plane labelled $0, 1, 2, \ldots, n$ which will be referred to as agents. Let $x_0, x_1, \ldots, x_n$ denote the coordinate vector of current positions of agents $0, 1, 2, \ldots n$ respectively with respect to a common frame of reference. Assume that the formation is suppose to come to rest and moreover that agents $1, 2, 3, \ldots, n$ are already at their proper positions in the formation and are at rest. Thus

$$\dot{x}_i = 0, \quad i \in \{1, 2, 3, \ldots, n\} \tag{1}$$

We further assume that the nominal model for how agent 0 moves is a kinematic point model of the form

$$\dot{x}_0 = u \tag{2}$$

where u is an open loop control taking values in $\mathbb{R}^2$.

Suppose that agent 0 can sense its distances $y_1, y_2, y_3, \ldots, y_n$ from neighboring agents $1, 2, 3, \ldots, n$ with uniformly bounded, additive errors $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ respectively. Thus

$$y_i = ||x_i - x_0|| + \epsilon_i, \quad i \in \{1, 2, \ldots, n\} \tag{3}$$

Suppose in addition that agent 0 is given a set of non-negative numbers $d_1, d_2, \ldots, d_n$, where d_i represents a desired distance from agent 0 to agent i. The problem is to devise a control law depending on the d_i and the y_i which, were the ϵ_i all zero, would cause agent 0 to move to a position in the formation which, for $i \in \{1, 2, \ldots, n\}$, is d_i units from agent i. We call this the n *neighbor station keeping problem.* We shall also require the controllers we devise to guarantee that errors between the y_i and their desired values eventually become small if the measurement errors are all small.

Let x^* denote the target position to which agent 0 would have to move were the station keeping problem solvable. Then x^* would have to satisfy

$$d_i = ||x_i - x^*||, \quad i \in \{1, 2, \ldots, n\} \tag{4}$$

There are two cases to consider:

1. If $n = 2$, there will be two solutions x^* to (4) if $|d_1 - d_2| < ||x_1 - x_2|| < d_1 + d_2$ and no solutions if either $|d_1 - d_2| > ||x_1 - x_2||$ or $||x_1 - x_2|| > d_1 + d_2$. We will assume that two solutions exist and that the target position is the one closest to the initial position of agent zero.

2. If $n \geq 3$, there will exist a solution x^* to (4) only if agents 1 through n are aligned in such a way so that the circles centered at the x_i of radii d_i all intersect at at least one point. If the x_i are so aligned and at least three x_i are not co-linear, then x^* is even unique. Such alignments are of course exceptional. To account for the more realistic situation when points are out of alignment, we will assume instead of (4), that there is a value of x^* for which

$$d_i = ||x^* - x_i|| + \bar{\epsilon}_i, \quad i \in \{1, 2, \ldots, n\} \tag{5}$$

   where each $\bar{\epsilon}_i$ is a small miss-alignment error.

Our specific control objective can now be stated. Devise a feedback control for agent 0, using the d_i and measurements y_i, which bounds the induced $\mathcal{L}^2$ gains from each ϵ_i and each $\bar{\epsilon}_i$ to each of the errors

$$e_i = y_i^2 - d_i^2, \quad i \in \{1, 2, 3, \ldots, n\} \tag{6}$$

We will address this problem using well known concepts and constructions from adaptive control.

3 Error Models

The controllers which we propose to study will all be based on suitably defined error models. We now proceed to develop these models.

3.1 Error Equations

To begin, we want to derive a useful expression for each e_i. In view of (3)

$$y_i^2 = ||x_i - x_0||^2 + 2\epsilon_i||x_i - x_0|| + \epsilon_i^2$$

But

$$||x_i - x_0||^2 = ||x_i - x^*||^2 + 2(x^* - x_i)'\bar{x}_0 + ||\bar{x}_0||^2$$

where

$$\bar{x}_0 = x_0 - x^* \tag{7}$$

Moreover from (5)

$$d_i^2 = ||x_i - x^*||^2 + 2\bar{\epsilon}_i||x_i - x^*|| + \bar{\epsilon}_i{}^2$$

From these expressions and the definition of e_i in (6) it follows that

$$e_i = 2(x^* - x_i)'\bar{x}_0 + ||\bar{x}_0||^2 + 2\epsilon_i||\bar{x}_0|| + \eta_i \tag{8}$$

where

$$\eta_i = 2\epsilon_i||x_i - x_0|| + \epsilon_i^2 - 2\bar{\epsilon}_i||x_i - x^*|| - \bar{\epsilon}_i^2 - 2\epsilon_i||\bar{x}_0||$$

Note that $|||x_i - x_0|| - ||\bar{x}_0||| \leq ||x_i - x^*||$ because of the triangle inequality and the definition of $\bar{x}_0$ in (7). From this and (5) it is easy to see that

$$|\eta_i| \leq (|\epsilon_i| + |\bar{\epsilon}_i|)\gamma_i \tag{9}$$

where $\gamma_i = 2d_i + |\epsilon_i - \bar{\epsilon}_i|$.

3.2 Station Keeping with $n = 3$ Neighbors

In this section we consider the case when $n = 3$. We shall assume that x_1, x_2, and x_3 are not co-linear. Note first that we can write

$$\dot{\bar{x}}_0 = u \tag{10}$$

because of (2) and the fact that $\bar{x}_0 = x_0 - x^*$. Let

$$e = \begin{bmatrix} e_1 - e_3 \\ e_2 - e_3 \end{bmatrix}$$

and define $q = B\bar{x}_0$, where

$$B = 2\,[\,x_3 - x_1 \quad x_3 - x_2\,]' \tag{11}$$

The error model for this case is then

$$e = q + \epsilon||B^{-1}q|| + \eta \tag{12}$$

$$\dot{q} = Bu \tag{13}$$

where

$$\epsilon = 2\begin{bmatrix} \epsilon_1 - \epsilon_3 \\ \epsilon_2 - \epsilon_3 \end{bmatrix} \qquad \eta = \begin{bmatrix} \eta_1 - \eta_3 \\ \eta_2 - \eta_3 \end{bmatrix}$$

Our assumption that the x_i are not co-linear implies that B is non-singular. Note that since B is nonsingular, $x_0 = x^*$ whenever $q = 0$. This in turn will be the case when $e = 0$ provided $\epsilon = 0$ and $\eta = 0$. The term $||B^{-1}q||\epsilon$ can be regarded as a perturbation and can be dealt with using standard small gain arguments. Essentially linear error models like (12), (13) can also be derived for any $n > 3$.

3.3 Station Keeping with $n = 2$ Neighbors

In the two-neighbor case we've assumed that $|d_1 - d_2| < ||x_1 - x_2|| < d_1 + d_2$ and thus that two solutions x^* to (4) exist. We will assume that $\bar{x}_0$ has been defined so that $||\bar{x}_0(0)||$ is the smaller of the two possibilities. As before, and for the same reason, (10) holds. For this version of the problem we define

$$e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

Let $q = B\bar{x}_0$, where now

$$B = 2\,[\,x^* - x_1 \quad x^* - x_2\,]' \tag{14}$$

The error model for this case is then

$$e = q + \epsilon||B^{-1}q|| + ||B^{-1}q||^2\mathbf{1} + \eta \tag{15}$$

$$\dot{q} = Bu \tag{16}$$

where

$$\mathbf{1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \epsilon = 2\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} \qquad \eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$$

Note that our assumption that $|d_1 - d_2| < ||x_1 - x_2|| < d_1 + d_2$ implies that x_1, x_2, x^* are not co-linear. This in turn implies that B is non-singular. The essential difference between this error model and the error model for the three neighbor case is that the two-neighbor agent model has a quadratic function of state in its readout equation whereas the three-neighbor error model does not.

4 Station Keeping Supervisory Controller

In this section we will develop a set of controller equations aimed at solving the station keeping problem with three neighbors. Because of its properties, the controller we propose can also be used for the two neighbor version of the problem; however in this case meaningful results can only be claimed if agent 0 starts out at a position which is sufficiently close to its target x^*. For ease of reference, we repeat the error equations of interest.

$$e = q + \epsilon||B^{-1}q|| + \eta \quad (17)$$

$$\dot{q} = Bu \quad (18)$$

In the sequel we will assume that $||\epsilon|| \leq \epsilon^*, \;\; t \geq 0$ where ϵ^* is a positive constant which satisfies the constraint

$$\epsilon^* < \frac{1}{||B^{-1}||} \quad (19)$$

Note that this constraint says that the allowable measurement error bound will decrease as agents 1, 2, and 3 are positioned closer and closer to co-linear and/or further and further away from agent 0. While we are unable to fully justify this assumption at this time, we suspect that it is intrinsic and is not specific to the particular approach to station keeping which we are following. Our suspicion is prompted in part by the observation that the map $q \longmapsto q + \epsilon||B^{-1}q||$ will be invertible for all $||\epsilon|| \leq \epsilon^*$ if and only if (19) holds.

The type of control system we intend to develop assumes that B is unknown, but requires one to define at the outset a closed bounded subset of 2×2 non-singular matrices $\mathcal{P} \subset \mathbb{R}^{2\times 2}$ which is big enough so that it can be assumed that $B \in \mathcal{P}$. $\mathcal{P}$ can consist of one connected subset or a finite union of compact, connected subsets. It is not necessary for the subsets to be disjoint. These properties can be used to advantage in defining $\mathcal{P}$. More about this later.

The supervisory control system to be considered consists of a "multi-estimator" $\mathbb{E}$, a "multi-controller" $\mathbb{C}$, a "monitor" $\mathbb{M}$ and a "dwell-time switching logic" $\mathbb{S}$. These terms and definitions have been discussed before in [6, 7] and elsewhere. They are fairly general concepts, have specific meanings, and apply to a broad range of problems. Although there is considerable flexibility in how one might define these component subsystems, in this paper we shall be quite specific. The numbered equations which follow, are the equations which define the supervisory controller we will consider.

4.1 Multi-Estimator $\mathbb{E}$

For the problem of interest, the multi-estimator $\mathbb{E}$ is defined by the two equations

$$\dot{z}_1 = -\lambda z_1 + \lambda e \quad (20)$$

$$\dot{z}_2 = -\lambda z_2 + u \quad (21)$$

where λ is a design constant which must be positive but is otherwise unconstrained.

Note that the signal $\rho = z_1 + Bz_2 - q$ satisfies

$$\dot{\rho} = -\lambda\rho + \lambda(\epsilon||B^{-1}q|| + \eta)$$

and that the *output estimation error*

$$\bar{e}_B = z_1 + Bz_2 - e$$

can be written as $\bar{e}_B = \rho - \epsilon||B^{-1}q|| - \eta$. These relationships can be conveniently represented by the block diagram in Figure 1. The diagram describes a nonlinear dynamical system with inputs η

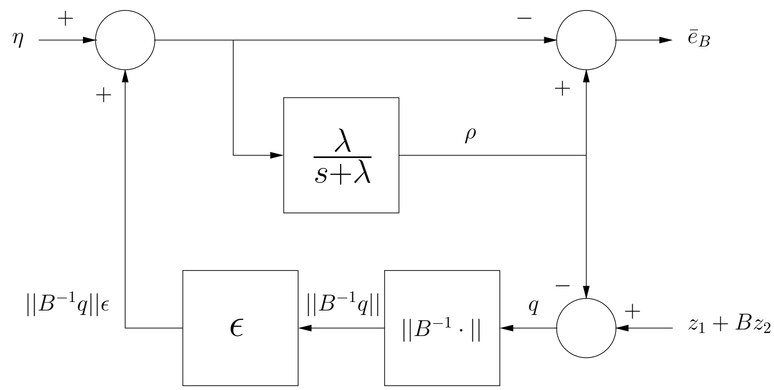

Figure 1: Subsystem

and $z_1 + Bz_2$ and output $\bar{e}_B$. It is easy to verify that this system is globally exponentially stable with stability margin no smaller than $\lambda(1 - \epsilon^*||B^{-1}||)$ because of the measurement constraint (19) discussed earlier. The diagram clearly implies that if ϵ and η were to tend to 0, so would $\bar{e}_B$; in this case $z_1 + Bz_2$ would therefore be an asymptotically correct estimate of e . We exploit these observations below.

4.2 Multi-Controller $\mathbb{C}$

The multi-controller $\mathbb{C}$ we propose to study is simply

$$u = -\lambda\widehat{B}^{-1}e \tag{22}$$

where $\widehat{B}$ is a piecewise constant switching signal taking values in $\mathcal{P}$. The definition of u has been crafted so that the "closed-loop parameterized system" matrix $-\lambda PP^{-1}$ is stable with "stability margin" λ for all $P \in \mathcal{P}$. Other controllers which accomplish this could also be used {e.g., $u = -\lambda\widehat{B}^{-1}(z_1+\widehat{B}z_2)$}. The consequence of this definition of u is predicted by the certainty equivalence stabilization theorem [8] and is as follows. Let $\bar{e}_{\widehat{B}} = z_1 + \widehat{B}z_2 - e$ and define the so called *injected sub-system* to be the system which results when $z_1 + Bz_2 - \bar{e}_{\widehat{B}}$ is substituted for e in the closed loop system determined by (20), (21) and (22). Thus

$$\begin{aligned}\dot{z}_1 &= \lambda\widehat{B}z_2 - \lambda\bar{e}_{\widehat{B}} \\ \dot{z}_2 &= -\lambda\widehat{B}^{-1}z_1 - 2\lambda z_2 + \lambda\widehat{B}^{-1}\bar{e}_{\widehat{B}}\end{aligned}$$

Certainty equivalence implies that this model, viewed as a system with input $\bar{e}_{\widehat{B}}$, is also stable with stability margin λ for each fixed $\widehat{B} \in \mathcal{P}$. In this special case one can deduce this directly using the state transformation $\{z_1, z_2\} \longmapsto \{z_1, z_1 + \widehat{B}z_2\}$. For this system to have stability margin λ means that for any positive number $\lambda_0 < \lambda$ the matrix $\lambda_0 I + A(\widehat{B})$ is exponentially stable for all constant $\widehat{B} \in \mathcal{P}$. Here

$$A(\widehat{B}) = \begin{bmatrix} 0 & \lambda\widehat{B} \\ -\lambda\widehat{B}^{-1} & (\lambda_0 - 2\lambda)I \end{bmatrix}$$

which is the state coefficient matrix of the injected system.

In the sequel, we fix λ_0 at any positive value such that $\lambda_0 < \lambda(1-\epsilon^*)||B||^{-1}$. This number turns out to be a lower bound on the convergence rate for the entire closed-loop control system.

We need to pick one more positive design parameter, called a *dwell time* τ_D. This number has to be chosen large enough so that the injected linear system defined above is exponentially stable with stability margin λ for every "admissible" piecewise constant switching signal $\widehat{B} : [0,\infty) \to \mathcal{P}$, where by *admissible* we mean a piecewise constant signal whose switching instants are separated by at least τ_D time units. This is easily accomplished because each $\lambda_0 I + A(P)$, $P \in \mathcal{P}$ is a stability matrix. All that's required then is to pick τ_D large enough so that the induced norm {any matrix norm} of each matrix $e^{\{\lambda_0 I + A(P)\}t}$, $P \in \mathcal{P}$, is less than 1.

It is useful for analysis to add to Figure 1, two copies of the injected system just defined, one $\{\Sigma_1\}$ with output $e = z_1 + \widehat{B}z_2 - \bar{e}_{\widehat{B}}$ and the other $\{\Sigma_2\}$ with output $z_1 + Bz_2$. The multiple copies are valid because the injected system each is an exponentially stable linear system. The resulting system is shown in Figure 2.

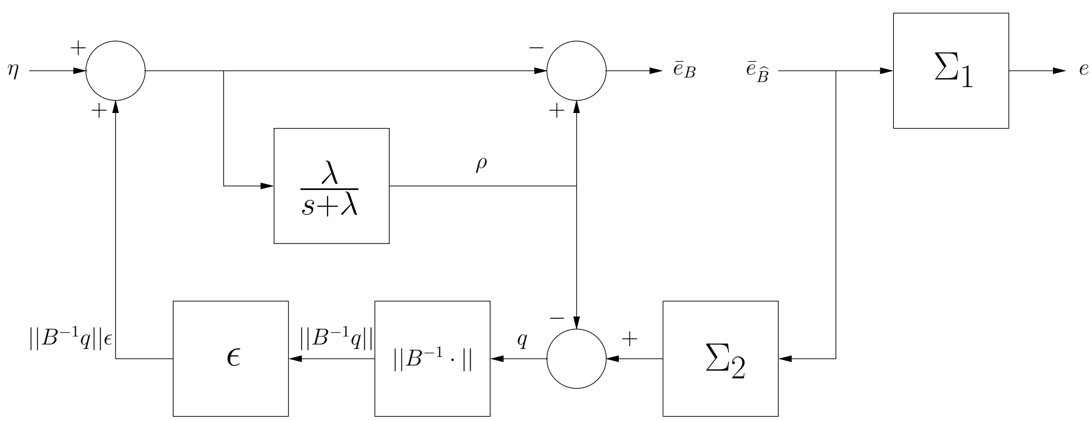

Figure 2: Subsystem for Analysis

Note that if there were a gain between $\bar{e}_B$ and $\bar{e}_{\widehat{B}}$, and if ϵ were small enough, the resulting system would be exponentially stable and bounded η would produce bounded e. We return to this observation later.

4.3 Monitor $\mathbb{M}$

The state dynamic of monitor $\mathbb{M}$ is defined by the equation

$$\dot{W} = -2\lambda_0 W + \begin{bmatrix} \lambda z_1 - e \\ z_2 \end{bmatrix} \begin{bmatrix} \lambda z_1 - e \\ z_2 \end{bmatrix}' \tag{23}$$

where W is a "weighting matrix" which takes values in the linear space $\mathcal{X}$ of 4×4 symmetric matrices. Note that it takes only 10 differential equations rather than 16 to generate W because of symmetry. The output of $\mathbb{M}$ is a parameter dependent "monitoring signal" $\mu_P = M(W, P)$ where $M : \mathcal{X} \times \mathcal{P} \rightarrow \mathbb{R}$ is defined as

$$M(X, P) = \text{trace}\{[\, I \quad P\,]\, X\, [\, I \quad P\,]'\} \tag{24}$$

The readout map $M(\cdot)$ is used in defining the switching logic $\mathbb{S}$. The signals $\mu_P,\ P \in \mathcal{P}$ are helpful in motivating the definition of $\mathbb{M}$ and the switching logic $\mathbb{S}$ which follows; however, they are actually not used anywhere in the implemented system. It is obvious that they could not be because there are infinitely many of them.

Note that for any $P \in \mathcal{P}$,

$$\dot{\mu}_P = -2\lambda_0 \mu_P + \text{trace}([\, \lambda z_1 + P z_2 - e\,]\, [\, \lambda z_1 + P z_2 - e\,]')$$

so

$$\dot{\mu}_P = -2\lambda_0 \mu_P + ||z_1 + P z_2 - e||^2$$

But $\bar{e}_P = z_1 + P z_2 - e$, so

$$\dot{\mu}_P = -2\lambda_0 \mu_P + ||\bar{e}_P||^2$$

Therefore, if for motivational purposes we were to temporarily initialize $W(0) = 0$, then

$$M(W, P) = \int_0^t \{e^{-2\lambda_0 (t-s)} ||\bar{e}_P||^2\} ds$$

Thus if we introduce the exponentially weighted 2 norm

$$||\omega||_t = \sqrt{\int_0^t \{e^{\lambda_0 s} ||\omega(s)||\}^2 ds}$$

where ω is a piecewise continuous signal, then

$$M(W(t), P) = e^{-2\lambda_0 t} ||\bar{e}_P||_t^2, \quad t \geq 0$$

Minimizing $M(W(t), P)$ with respect to P and setting $\widehat{B}(t)$ equal to the minimizing value, would then yield an inequality of the form

$$||\bar{e}_{\widehat{B}}||_t \leq ||\bar{e}_B||_t$$

Were it possible to accomplish this at every instant of time and were $\widehat{B}$ changing slowly enough so that all of the time-varying subsystems in Figure 2 were exponentially stable, then one could conclude that for ϵ^* sufficiently small, the resulting overall system with input η and output e would be stable with respect to the exponentially weighted norm we've been discussing. It is of course not possible to carry out these steps instantly and even if it were, $\widehat{B}$ would likely be changing too fast for the time-varying subsystems in Figure 2 to be exponentially stable. What will be achieved is not quite this because of the requirement that $\widehat{B}$ not change too fast. Nonetheless, we will end up with an input-output stable system.

4.4 Dwell-time Switching Logic $\mathbb{S}$

For our purposes a *dwell-time switching logic* $\mathbb{S}$, is a hybrid dynamical system whose input and output are W and $\widehat{B}$ respectively, and whose state is the ordered triple $\{X, \tau, \widehat{B}\}$. Here X is a discrete-time matrix which takes on sampled values of W, and τ is a continuous-time variable called a *timing signal*. τ takes values in the closed interval $[0, \tau_D]$. Also assumed pre-specified is a *computation time* $\tau_C \leq \tau_D$ which bounds from above for any $X \in \mathcal{W}$, the time it would take a supervisor to compute a value $P \in \mathcal{P}$ which minimizes $M(X, B)$. Between "event times," τ is generated by a reset integrator according to the rule $\dot{\tau} = 1$. Event times occur when the value of τ reaches either $\tau_D - \tau_C$ or τ_D; at such times τ is reset to either 0 or $\tau_D - \tau_C$ depending on the value of $\mathbb{S}$'s state. $\mathbb{S}$'s internal logic is defined by the flow diagram shown in Figure 3 where P_X denotes a value of $P \in \mathcal{P}$ which minimizes $M(X, P)$.

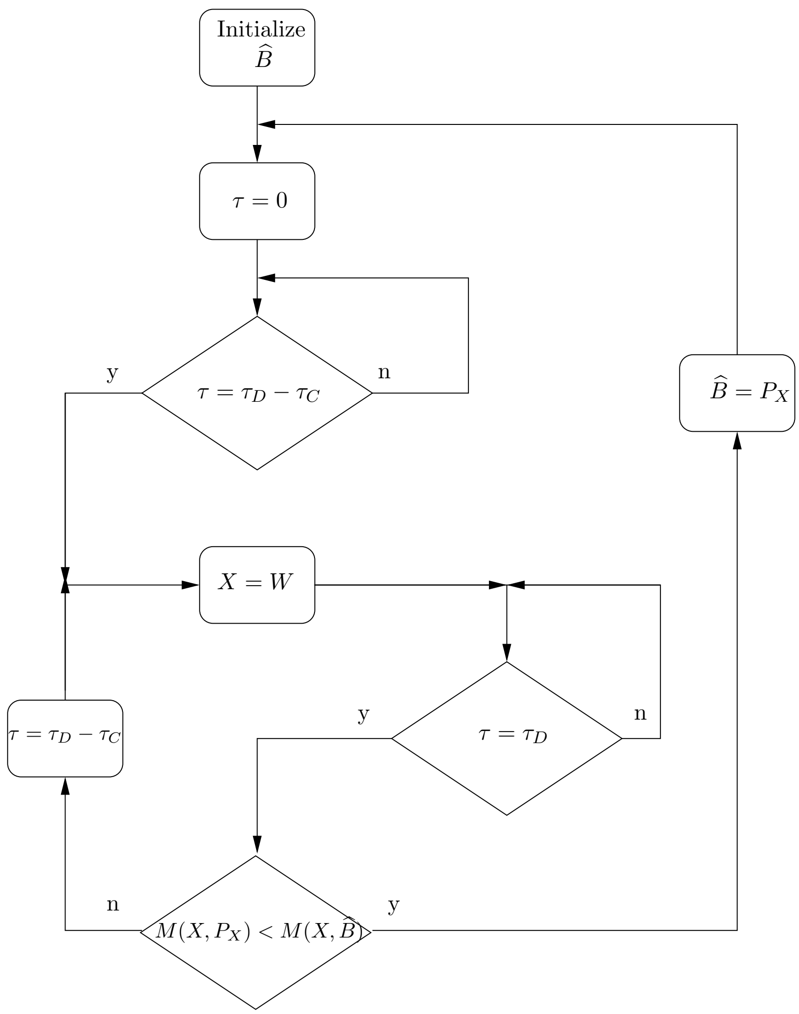

Figure 3: Dwell-Time Switching Logic $\mathbb{S}$

The definition of $\mathbb{S}$ clearly implies that its output $\widehat{B}$ is an admissible switching signal. This means that switching cannot occur infinitely fast and thus that existence and uniqueness of solutions to the differential equations involved is not an issue.

Note that implementation of the switching logic just described requires an algorithm capable of minimizing $M(X,P)$ over $\mathcal{P}$ for various values of $X \in \mathcal{X}$. Although the quadratic term in $M(X,P)$ is a positive semi-definite function in the elements of P and $\mathcal{P}$ is compact, this minimization problem is nonetheless formidable because $\mathcal{P}$ is typically not a convex set or even a finite union of convex sets. While this issue does not in any way limit the theoretical validity of the algorithm we are discussing, it is of obvious practical importance when implementation is taken into account. There are several different ways one might seek to deal with this issue. We will discuss each of them later in the paper.

5 Results

The results which follow rely heavily on the following Proposition which characterizes the effect of the monitor-dwell time switching logic subsystem.

Proposition 1 *Suppose that $\mathcal{P}$ is a compact subset of a finite dimensional space, that $W(0) = 0$, that $\widehat{B}$ is the response of the monitor-switching logic subsystem $\{\mathbb{M}, \mathbb{S}\}$ to any continuous input signals e, z_1, and z_2 taking values in $\mathbb{R}^2$, and that $\bar{e}_P = z_1 + Pz_2 - e,\ P \in \mathcal{P}$. For each real number $\gamma > 0$ and each fixed time $T > 0$, there exists piecewise-constant signals $H : [0,\infty) \to \mathbb{R}^{2\times 4}$ and $\psi : [0,\infty) \to \{0,1\}$ such that*

$$|H(t)| \leq \gamma, \quad t \geq 0 \tag{25}$$

$$\int_0^\infty \psi(t)dt \leq 4(\tau_D + \tau_C) \tag{26}$$

and

$$||(1-\psi)(\bar{e}_{\widehat{B}} - Hz) + \psi \bar{e}_B||_T \leq \delta ||\bar{e}_B||_T \tag{27}$$

where

$$\delta = 1 + 8\left(\frac{1 + \text{diameter}\{\mathcal{P}\}}{\gamma}\right)^4$$

and $z = [\, z_1' \quad z_2' \,]'$.

This proposition is proved in [6, 7]. The proposition summarizes the key consequences of dwell time switching which are needed to analyze the system under consideration. While the inequality in (27) is more involved than the inequality $||\bar{e}_{\widehat{B}}||_t \leq ||\bar{e}_B||_t$ mentioned earlier, the former is provably correct whereas the latter is not. Despite its complexity, (27) can be used to establish input-output stability with respect to the exponentially weighted norm $||\cdot||_t$. The idea is roughly as follows. Fix $T > 0$ and pick γ small enough so that $\lambda_0 I + A(\widehat{B}) + (1-\psi)D(\widehat{B})H$ is exponentially stable where $D(\widehat{B}) = [\, -\lambda I' \quad \lambda(\widehat{B}^{-1})' \,]'$. The fact that ψ has a finite $\mathcal{L}^1$ norm {cf. (26)}, implies that $\lambda_0 I + A(\widehat{B}) + (1-\psi)D(\widehat{B})H + \psi\,[\, 0 \quad \widehat{B} - B \,]$ is exponentially stable as well. Next define

$$\bar{e} = (1-\psi)(\bar{e}_{\widehat{B}} - Hz) + \psi \bar{e}_B$$

Then

$$||\bar{e}||_T \leq \delta ||\bar{e}_B||_T \tag{28}$$

because of (27). The definition of $\bar{e}$ implies that

$$\bar{e}_{\widehat{B}} = \bar{e} + (1-\psi)Hz + \psi\,[\,0 \quad \widehat{B} - B\,]\,z$$

Substitution into the injected system defined earlier yields the exponentially stable system

$$\dot{z} = \{A(\widehat{B}) + (1-\psi)D(\widehat{B})H + \psi\,[\,0 \quad \widehat{B} - B\,]\}z + D(\widehat{B})\bar{e}$$

with input $\bar{e}$. Now add to Figure 1, two copies of the system just defined, one $\{\bar{\Sigma}_1\}$ with output $e = [\,I \quad \widehat{B}\,]\,z - \{\bar{e} + (1-\psi)Hz + \psi\,[\,0 \quad \widehat{B} - B\,]\,z\}$ and the other $\{\bar{\Sigma}_2\}$ with output $z_1 + Bz_2 = [\,I \quad B\,]\,z$. Like before, the multiple copies are valid because the matrix $A(\widehat{B}) + (1-\psi)D(\widehat{B})H + \psi\,[\,0 \quad \widehat{B} - B\,]$ is exponentially stable. The resulting system is shown in Figure 4. In the light of (28) it is easy to

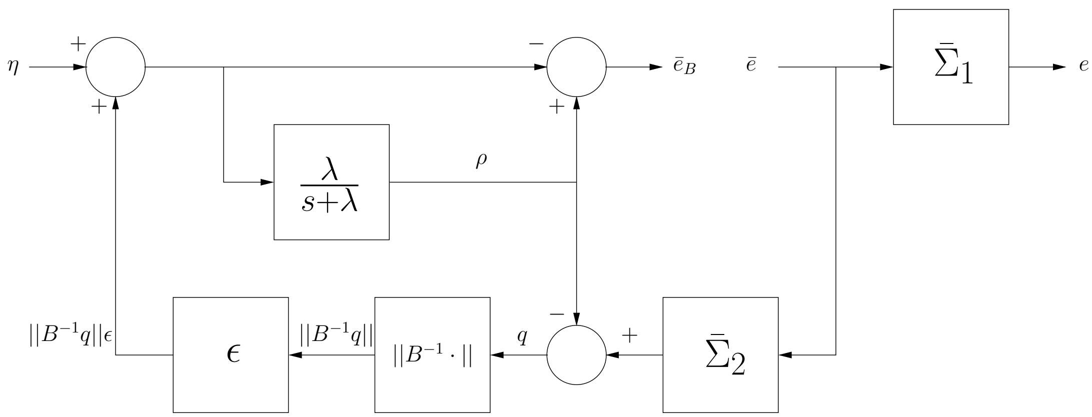

Figure 4: Snapshot at time T of the Overall Subsystem for Analysis

see that if the bound ϵ^* on ϵ is sufficiently small, the induced gain of this system from η to e with respect to $||\cdot||_T$ is bounded by a finite constant g_T. It can be shown that g_T in turn, is bounded above by a constant g not depending on T [7]. Since this is true for all T, it must be true that g bounds the induced gain from η to e with respect to $||\cdot||_\infty$.

The following results are fairly straightforward consequences of these ideas. Detailed proofs, specific to the problem at hand, can be found in the full-length version of this paper. The results are as follows:

1. If all measurement errors ϵ_i and all miss-alignment errors $\bar{e}_i$ are zero, then, no matter what its initial value, $x_0(t)$ tends to the unique solution x^* to (4) as fast as $e^{-\lambda_0 t}$.

2. If the measurement errors ϵ_i and the miss-alignment errors $\bar{e}_i$ are not all zero, and the ϵ_i sufficiently small, then no matter what its initial value, $x_0(t)$ tends to a value for which the norm of the error e is bounded by a constant times the sum of the norms of the ϵ_i and the $\bar{e}_i$.

6 Dealing with a Non-Convex Parameter Space

Although the quadratic term in $M(X, P)$ is a positive semi-definite function of the elements of P, the problem of minimizing $M(X, P)$ over $\mathcal{P}$ is still very complex because $\mathcal{P}$ is not typically convex or even a finite union of convex sets. The root of the problem stems from the requirement that the algebraic curve

$$\mathcal{S} = \{P : p_{11}p_{22} - p_{12}p_{21} = 0\}$$

in $\mathbb{R}^{2\times 2}$ on which P is singular cannot intersect $\mathcal{P}$. There is considerable experience with simulations which suggests that this singularity issue can simply be ignored, because the chances of encountering a minimizing P which lies in $\mathcal{S}$ are very low. Nonetheless one would like to have a systematic way of dealing with this problem. One such approach relies on an idea called "cyclic switching" which was specifically devised to deal with this type of problem [9, 10]. Cyclic switching is roughly as follows. First $\mathcal{P}$ is allowed to contain singular matrices, in which case it is reasonable to assume that it is a finite union of compact convex sets. Minimization over $\mathcal{P}$ thus becomes a finite number of standard quadratic programming problems. For minimizing values of $\widehat{B}$ which turn out to be close to or on $\mathcal{S}$, one uses a specially structured switching controller in place of (22) – one which does not require $\widehat{B}$ to be nonsingular. This controller is used for a specific length of time over which a "switching cycle" takes place. At the end of the cycle, minimization of $M(W, \widehat{B})$ is again carried out; if $\widehat{B}$ is again close to $\mathcal{S}$, another switching cycle is executed. On the other hand, if $\widehat{B}$ is not close to $\mathcal{S}$, the standard certainty equivalence control (22) is used. Cyclic switching is completely systematic and can be shown to solve the singularity problem of interest here. The main disadvantage of cyclic switching is that it introduces additional complexity. This matter will considered in detail in a future paper.

There is another possible way to deal with the singularity problem. What we'd really like is to construct a parameter space $\mathcal{P}$ which is a finite union of convex sets, defined so that every matrix in $\mathcal{P}$ is nonsingular and, in addition, the matrices in $\mathcal{P}$ correspond to a "large" class of possible positions of agents $1, 2, 3$. Keep in mind that the convex subsets whose union defines such a $\mathcal{P}$, *can* overlap. This suggests the following problem.

Convex Covering Problem: Suppose that we are given a compact subset $\mathcal{P}_0$ of a finite dimensional space which is disjoint from a second closed subset $\mathcal{S}$ {typically an algebraic curve}. Define a *convex cover* of $\mathcal{P}_0$ to mean a finite set of possibly overlapping convex subsets $\mathcal{E}_i$ such that the union of the $\mathcal{E}_i$ contains $\mathcal{P}_0$ but is disjoint from $\mathcal{S}$. One could then define $\mathcal{P}$ to be the union of the $\mathcal{E}_i$. To the best of our knowledge, this is an open problem. Its solution would solve the singularity problem we've been discussing.

7 Concluding Remarks

In this paper we have devised a constructive solution to the three neighbor station keeping problem in which range measurements are the only sensed signals upon which station keeping is to be based. The solution is provably correct and the performance of the resulting system degrades gracefully in the face of measurement and miss-alignment errors, provided the measurement errors are not too large. We have used standard constructions from adaptive control to accomplish this. Because of the exponential stability of the overall system, the same control algorithm will solve the two agent station keeping problem provided the agent is initially not too far from its target position.

We have so far performed many simulations and have observed results consistent with our theoretical findings. In the very near future we plan to carry out experiments on a mobile robot testbed.

The same approach followed in this paper can be used to address the problem of maintaining an agent's position in a *moving* formation in the plane using range-only measurements. This generalization is developed in [11].

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