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Sensor network localization with imprecise distances

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Abstract

An approach to formulate geometric relations among distances between nodes as equality constraints is introduced in this paper to study the localization problem with imprecise distance information in sensor networks. These constraints can be further used to formulate optimization problems for distance estimation. The optimization solutions correspond to a set of distances that are consistent with the fact that sensor nodes live in the same plane or 3D space as the anchor nodes. These techniques serve as the foundation for most of the existing localization algorithms that depend on the sensors' distances to anchors to compute each sensor's location.

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1. Introduction

In sensor networks, sensors' location information is vital in location-aware applications and influences network performances when algorithms like geographic routing are used. Hence, localization is crucial in the development of low-cost sensor networks where it is not feasible for all the nodes' locations to be directly measurable via GPS or other similar means. The locations of some of the nodes have to be inferred by utilizing estimated distances to their nearby nodes. Hence, as pointed out in [7], the first and the most fundamental phase of most localization algorithms [9–11] is the determination of the distances between sensor nodes whose locations are to

be computed and “anchor nodes”. An *anchor node* is a node whose location is assumed to be known during the current computation. An anchor node may be a node with a GPS device, or a node with a tentative estimated location in an iterative computation process [11], or a point in the trajectory of a mobile beacon [13], etc. Distances between sensors and anchors can be obtained via direct measurements if they are within sensing range of one another; otherwise, approximation methods such as sum-dist [7,11] and DV-hop [9,10] can be used to estimate the sensor–anchor distances. No matter which method is used to obtain the distances, the data acquired are usually imprecise compared with the true distances because of measurement noise and estimation errors. Because the true distances between nodes are interdependent, these inaccuracies have undesirable consequences of causing inconsistency with respect to geometric relations, and sometimes may even cause localization algorithms to collapse. For example, in a 2D scenario, triangulation fails due to nonexistence of feasible solutions when the distances are not consistent with the fact that all sensors live on a plane.

However, because there are geometric and algebraic relations among the true distances between the nodes, the errors associated with these imprecise distances are not independent. It follows that one can exploit this dependence by seeking, for

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example, to find that set of errors which would yield true distances as close as possible (in some metric) to the measured imprecise distances. These errors are then associated with the imprecise distances to correct those distances and use the corrected values to provide a nominal location for the sensor whose position is to be determined, by iteration [9,10], min–max optimization [11], or any of the other appropriate methods.

In this paper, a novel technique is presented which describes the geometric relations among the sensor–anchor distances as one or multiple quadratic equality constraints. The key step is to use the *Cayley–Menger determinant* that is defined in the following section. Although inequality constraints usually variations of the triangle inequality where the sum of the lengths of two sides of the triangle must be greater than the third have been discussed before [15], the equality constraints reported in this paper disclose more insightfully the dependency among distances between nodes. Furthermore, these equality constraints can be utilized to estimate errors in the measured or computed distances. One specific estimation approach presented in this paper is to solve a least squares problem where the objective is to minimize the sum of the squared errors in the distances that are measured or computed by a sensor.

The rest of the paper is organized as follows. In Section 2, the definition of the Cayley–Menger determinant is introduced as well as related classic results in Distance Geometry. In Section 3, geometric relations among nodes’ positions are formulated as quadratic constraints by using the Cayley–Menger determinant. In Section 4, we show that errors in the imprecise distance information can be estimated by solving an optimization problem. In Section 5, we provide a computational example.

2. Cayley–Menger determinants

The *Cayley–Menger matrix* of two sequences of n points, $\{\mathbf{p}_1, \dots, \mathbf{p}_n\}$ and $\{\mathbf{q}_1, \dots, \mathbf{q}_n\} \in \mathbb{R}^m$, is defined as

$$M(\mathbf{p}_1, \dots, \mathbf{p}_n; \mathbf{q}_1, \dots, \mathbf{q}_n) \triangleq \begin{bmatrix} d^2(\mathbf{p}_1, \mathbf{q}_1) & d^2(\mathbf{p}_1, \mathbf{q}_2) & \cdots & d^2(\mathbf{p}_1, \mathbf{q}_n) & 1 \\ d^2(\mathbf{p}_2, \mathbf{q}_1) & d^2(\mathbf{p}_2, \mathbf{q}_2) & \cdots & d^2(\mathbf{p}_2, \mathbf{q}_n) & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d^2(\mathbf{p}_n, \mathbf{q}_1) & d^2(\mathbf{p}_n, \mathbf{q}_2) & \cdots & d^2(\mathbf{p}_n, \mathbf{q}_n) & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}, \quad (1)$$

where $d(\mathbf{p}_i, \mathbf{q}_j)$, $i, j \in \{1, \dots, n\}$ is the Euclidean distance between the points \mathbf{p}_i and \mathbf{q}_j . The *Cayley–Menger bideterminant* [3] of these two sequences of n points is defined as

$$D(\mathbf{p}_1, \dots, \mathbf{p}_n; \mathbf{q}_1, \dots, \mathbf{q}_n) \triangleq \det M(\mathbf{p}_1, \dots, \mathbf{p}_n; \mathbf{q}_1, \dots, \mathbf{q}_n). \quad (2)$$

This determinant is widely used in distance geometry theory [1,3] which deals with Euclidean geometry in spaces where “distance” is defined and invariant. When the two sequences of points are the same, $M(\mathbf{p}_1, \dots, \mathbf{p}_n; \mathbf{p}_1, \dots, \mathbf{p}_n)$ and $D(\mathbf{p}_1, \dots, \mathbf{p}_n; \mathbf{p}_1, \dots, \mathbf{p}_n)$ are denoted for convenience by

$M(\mathbf{p}_1, \dots, \mathbf{p}_n)$ and $D(\mathbf{p}_1, \dots, \mathbf{p}_n)$ respectively, and the latter is simply called a *Cayley–Menger determinant*.

The Cayley–Menger determinant provides another way of expressing the hyper-volume of a “simplex” by using only the lengths of the edges. A *simplex* of n points is the smallest $(n - 1)$ -dimensional convex hull containing these points. The hyper-volume V of the simplex formed by the points $\mathbf{p}_1, \dots, \mathbf{p}_n$ is given by Crippen and Havel [3]

$$V^2(\mathbf{p}_1, \dots, \mathbf{p}_n) = \frac{(-1)^n}{2^{n-1}((n-1)!)^2} D(\mathbf{p}_1, \dots, \mathbf{p}_n). \quad (3)$$

We can check Eq. (3) for the following low-dimensional cases:

- For $n = 2$, $D(\mathbf{p}_1, \mathbf{p}_2) = 2d^2(\mathbf{p}_1, \mathbf{p}_2)$, and $V(\mathbf{p}_1, \mathbf{p}_2) = d(\mathbf{p}_1, \mathbf{p}_2)$.
- For $n = 3$, the simplex is the triangle formed by \mathbf{p}_1 , \mathbf{p}_2 , and \mathbf{p}_3 . Then $V(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ is the area of this triangle. Let a, b, c be the lengths of the three edges of the triangle, namely $a = d(\mathbf{p}_1, \mathbf{p}_2)$, $b = d(\mathbf{p}_2, \mathbf{p}_3)$, $c = d(\mathbf{p}_3, \mathbf{p}_1)$. Let s denote the semi-perimeter $s = \frac{1}{2}(a + b + c)$. Then from Heron’s formula [2], we know that $V(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = \sqrt{s(s-a)(s-b)(s-c)}$. Hence, it is easy to check that $V^2(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = (-1/16)D(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$.
- For $n = 4$, the simplex is the tetrahedron formed by \mathbf{p}_1 , \mathbf{p}_2 , \mathbf{p}_3 , and \mathbf{p}_4 . We can obtain Euler’s formula [14] relating the volume of a tetrahedron with its edge-lengths: $V^2(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = \frac{1}{288}D(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4)$.

The following theorem is a classical result on the Cayley–Menger determinant and is later generalized in [16].

Theorem 1. Consider an n -tuple of points $\mathbf{p}_1, \dots, \mathbf{p}_n$ in m -dimensional space. If $n \geq m + 2$, then the Cayley–Menger matrix $M(\mathbf{p}_1, \dots, \mathbf{p}_n)$ is singular, namely

$$D(\mathbf{p}_1, \dots, \mathbf{p}_n) = 0. \quad (4)$$

A stronger statement can be made as follows in terms of the rank of the Cayley–Menger matrix.

Theorem 2 (Theorem 112.1 in Blumenthal [1]). Consider an n -tuple of points $\mathbf{p}_1, \dots, \mathbf{p}_n$ in m -dimensional space with $n \geq m + 1$. The rank of the Cayley–Menger matrix $M(\mathbf{p}_1, \dots, \mathbf{p}_n)$ is at most $m + 1$.

In fact, the rank of $M(\mathbf{p}_1, \dots, \mathbf{p}_n)$ equals $m + 1$ if and only if at least $m + 1$ points of the n points are in generic positions. A similar statement made in terms of the cofactors of the Cayley–Menger determinant can be found in Corollary 1 of [16].

3. Geometric relations as equality constraints

In this section, we will illustrate how one can describe the geometric relations among the distances between nodes as algebraic constraints, which are, to be precise, quadratic algebraic

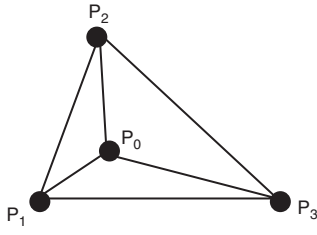


Fig. 1. Sensor node 0 with anchor nodes 1, 2 and 3 in 2D.

equations. We first consider the two-dimensional case, and the result will then be generalized to the three-dimensional case.

3.1. 2D Case

As shown in Fig. 1, nodes 1, 2 and 3 are anchor nodes in known positions and node 0 is a sensor node with unknown position to be localized. We assume that nodes 1, 2 and 3 are in noncollinear positions.

Let $d_{ij} = d(\mathbf{p}_i, \mathbf{p}_j)$ denote the accurate Euclidean distance between nodes i and j with $i, j=0, 1, 2, 3$. Now suppose that inaccurate distances \bar{d}_{0i} , $i=1, 2, 3$ acquired by either noisy range measurements or computations [9–11], are available while the d_{ij} , the accurate Euclidean distances between anchor nodes with $i \neq j$, $i, j = 1, 2, 3$, are known. Then

$$\bar{d}_{0i}^2 = d_{0i}^2 - \varepsilon_i \quad (5)$$

for some error ε_i .

Theorem 3. *The errors ε_i for $i=1, 2, 3$ as defined immediately above satisfy a single algebraic equality which is quadratic though not homogeneous in the ε_i 's, and whose coefficients are determined by \bar{d}_{0i} for $i=1, 2, 3$ and d_{ij} for $i, j=1, 2, 3$ and $i \neq j$:*

$$\varepsilon^T A \varepsilon + \varepsilon^T b + c = 0, \quad (6)$$

where

$$\varepsilon = [\varepsilon_1 \quad \varepsilon_2 \quad \varepsilon_3]^T, \quad (7)$$

$$A = \begin{bmatrix} 2d_{23}^2 & d_{12}^2 - d_{13}^2 - d_{23}^2 & d_{13}^2 - d_{23}^2 - d_{12}^2 \\ d_{12}^2 - d_{13}^2 - d_{23}^2 & 2d_{13}^2 & d_{23}^2 - d_{12}^2 - d_{13}^2 \\ d_{13}^2 - d_{12}^2 - d_{23}^2 & d_{23}^2 - d_{12}^2 - d_{13}^2 & 2d_{12}^2 \end{bmatrix}, \quad (8)$$

$$b_1 = 4d_{23}^2 \bar{d}_{01}^2 + 2(d_{12}^2 - d_{13}^2 - d_{23}^2) \bar{d}_{02}^2 + 2(d_{13}^2 - d_{12}^2 - d_{23}^2) \bar{d}_{03}^2 + 2d_{23}^2 (d_{23}^2 - d_{12}^2 - d_{13}^2), \quad (9)$$

$$b_2 = 4d_{13}^2 \bar{d}_{02}^2 + 2(d_{12}^2 - d_{13}^2 - d_{23}^2) \bar{d}_{01}^2 + 2(d_{23}^2 - d_{12}^2 - d_{13}^2) \bar{d}_{03}^2 + 2d_{13}^2 (d_{13}^2 - d_{12}^2 - d_{23}^2), \quad (10)$$

$$b_3 = 4d_{12}^2 \bar{d}_{03}^2 + 2(d_{13}^2 - d_{12}^2 - d_{23}^2) \bar{d}_{01}^2 + 2(d_{23}^2 - d_{12}^2 - d_{13}^2) \bar{d}_{02}^2 + 2d_{12}^2 (d_{12}^2 - d_{13}^2 - d_{23}^2), \quad (11)$$

$$c = 2d_{12}^2 d_{13}^2 d_{23}^2 + 2d_{23}^2 \bar{d}_{01}^4 + 2d_{13}^2 \bar{d}_{02}^4 + 2d_{12}^2 \bar{d}_{03}^4 + 2(d_{12}^2 - d_{13}^2 - d_{23}^2) \bar{d}_{01}^2 \bar{d}_{02}^2 + 2(d_{13}^2 - d_{12}^2 - d_{23}^2) \bar{d}_{01}^2 \bar{d}_{03}^2 + 2(d_{23}^2 - d_{12}^2 - d_{13}^2) \bar{d}_{02}^2 \bar{d}_{03}^2 + 2d_{23}^2 (d_{23}^2 - d_{12}^2 - d_{13}^2) \bar{d}_{01}^2 + 2d_{13}^2 (d_{13}^2 - d_{12}^2 - d_{23}^2) \bar{d}_{02}^2 + 2d_{12}^2 (d_{12}^2 - d_{13}^2 - d_{23}^2) \bar{d}_{03}^2. \quad (12)$$

Furthermore, the matrix A is positive semi-definite.

Proof. From Theorem 1 we know that $D(\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = 0$, namely

$$\det \begin{bmatrix} 0 & d_{01}^2 & d_{02}^2 & d_{03}^2 & 1 \\ d_{01}^2 & 0 & d_{12}^2 & d_{13}^2 & 1 \\ d_{02}^2 & d_{12}^2 & 0 & d_{23}^2 & 1 \\ d_{03}^2 & d_{13}^2 & d_{23}^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} = 0. \quad (13)$$

Since nodes 1, 2 and 3 are noncollinear, $D(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \neq 0$. So we can define

$$E \triangleq \begin{bmatrix} 0 & d_{12}^2 & d_{13}^2 & 1 \\ d_{12}^2 & 0 & d_{23}^2 & 1 \\ d_{13}^2 & d_{23}^2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}^{-1}. \quad (14)$$

Then from (13) we know that

$$[d_{01}^2 \quad d_{02}^2 \quad d_{03}^2 \quad 1] E \begin{bmatrix} d_{01}^2 \\ d_{02}^2 \\ d_{03}^2 \\ 1 \end{bmatrix} = 0. \quad (15)$$

Here we have used the fact that for an arbitrary matrix $B = \begin{bmatrix} b_{11} & \mathbf{b}_{12} \\ \mathbf{b}_{12}^T & B_{22} \end{bmatrix}$, where b_{11} is a scalar, if B_{22} is nonsingular, then

$$\det B = (b_{11} - \mathbf{b}_{12} B_{22}^{-1} \mathbf{b}_{12}^T) \det B_{22}.$$

It follows that

$$[\bar{d}_{01}^2 + \varepsilon_1 \quad \bar{d}_{02}^2 + \varepsilon_2 \quad \bar{d}_{03}^2 + \varepsilon_3 \quad 1] E \begin{bmatrix} \bar{d}_{01}^2 + \varepsilon_1 \\ \bar{d}_{02}^2 + \varepsilon_2 \\ \bar{d}_{03}^2 + \varepsilon_3 \\ 1 \end{bmatrix} = 0, \quad (16)$$

which defines a quadratic surface in the ε_i 's. Multiplying both sides of (16) by the determinant of the inverse of E , we arrive at (6).

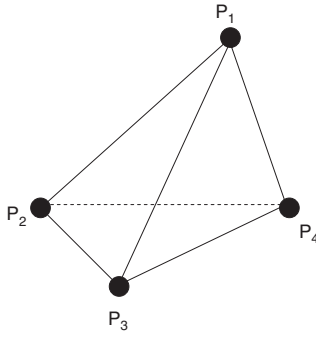


Fig. 2. Tetrahedron formed by four anchor nodes in 3D.

Let the coordinates of anchor i 's location be x_i and y_i with $i = 1, 2, 3$. Let

$$X = \begin{bmatrix} x_3 - x_2 & y_3 - y_2 \\ x_1 - x_3 & y_1 - y_3 \\ x_2 - x_1 & y_2 - y_1 \end{bmatrix}. \quad (17)$$

Then it is easily verified that

$$A = 2XX^T, \quad (18)$$

which is a positive semi-definite matrix. \square

If noisy distance information between sensor node 0 and r ($r > 3$) anchor nodes is available to sensor 0, we can write down $r - 2$ independent quadratic equality constraints. This can be obtained by demanding the coplanarity of the following node sets: $\{0, 1, 2, 3\}$, $\{0, 1, 2, 4\}$, \dots , $\{0, 1, 2, r\}$. Each coplanarity condition gives rise to a quadratic equality constraint in the following form through a procedure similar to that described above,

$$f_i(\varepsilon_1, \varepsilon_2, \varepsilon_i) = 0, \quad i = 3, 4, \dots, r. \quad (19)$$

Further coplanarity constraints can be written down using other selections for four nodes, e.g. $\{0, 2, 3, 4\}$, but such constraints will not be independent of the set in (19).

3.2. 3D Case

In 3D space, we consider the tetrahedron, as shown in Fig. 2, spanned by four anchor nodes 1, 2, 3 and 4, whose inaccurate distances relative to sensor node 0 are available to sensor 0. We assume the four anchor nodes are not in co-planar positions.

Similarly to the 2D case, we have

$$D(\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = 0, \quad (20)$$

which defines a quadratic surface in ε_i with $i = 1, 2, 3, 4$

$$\varepsilon^T \tilde{A} \varepsilon + \varepsilon^T \tilde{b} + \tilde{c} = 0. \quad (21)$$

After examining carefully each entry in matrix \tilde{A} , we define

$$\mathbf{Y} = \begin{bmatrix} (\mathbf{p}_2 - \mathbf{p}_4) \times (\mathbf{p}_3 - \mathbf{p}_4) \\ (\mathbf{p}_1 - \mathbf{p}_4) \times (\mathbf{p}_3 - \mathbf{p}_4) \\ (\mathbf{p}_1 - \mathbf{p}_4) \times (\mathbf{p}_2 - \mathbf{p}_4) \\ (\mathbf{p}_1 - \mathbf{p}_3) \times (\mathbf{p}_2 - \mathbf{p}_3) \end{bmatrix}^T, \quad (22)$$

which is a 4-by-3 matrix, where “ \times ” denotes the usual cross product of two vectors in 3D. Then

$$\tilde{A} = k\mathbf{Y} \cdot \mathbf{Y}^T, \quad (23)$$

where k is a nonzero scaling factor. Hence, the matrix \tilde{A} is also semi-definite.

Similar to the 2D case, if noisy distance information between sensor node 0 and s ($s > 4$) anchor nodes is available to sensor 0, we can write down $s - 3$ independent quadratic equality constraints.

4. An optimization problem

Given all the algebraic constraints obtained in the last section, we now try to estimate the error in the inaccurate distances between sensor nodes and anchor nodes. One approach is to formulate the problem as a least squares problem to minimize the sum of the squared errors. Other objective functions are also adoptable depending on the specific application context. As discussed in [8], the least squares approach is sometimes not the most appropriate one to use. We use it here simply because of its clarity and simplicity of expression. The main point is that the quadratic constraints, once established, can be a powerful tool in various applications such as least squares optimizations. Here we use the 2D case to illustrate the least squares approach while the 3D case can be dealt with by using the same procedures.

Let ε_i as defined in (5) be the error in the estimated squared distances between sensor 0 and anchor i . We want to minimize the sum of the squared errors

$$J = \varepsilon_1^2 + \varepsilon_2^2 + \dots + \varepsilon_r^2, \quad r \geq 3, \quad (24)$$

subject to $r - 2$ quadratic equality constraints as defined in Eq. (19).

When $r = 3$, we have a least squares problem with one quadratic constraint, which is well studied [4–6]. When $r > 3$, we can use the following Lagrangian multiplier method.

Let λ_i , $i = 1, \dots, r - 2$ be the Lagrangian multipliers. We can get the following Lagrangian multiplier form

$$\begin{aligned} H(\varepsilon_1, \dots, \varepsilon_r, \lambda_1, \dots, \lambda_{r-2}) \\ = \sum_{i=1}^r \varepsilon_i^2 + \sum_{i=1}^{r-2} \lambda_i f_{i+2}(\varepsilon_1, \varepsilon_2, \varepsilon_{i+2}). \end{aligned} \quad (25)$$

Because of the strict convexity of the function J and the positive semi-definiteness of the Hessians of functions f_{i+2} , when $\lambda_i > 0$, the Lagrangian H is a strictly convex function. Then

there exists a unique global minimum. So numerical methods, such as gradient methods, can be exploited to search for the minimum. In 3D, the same technique can be used to formulate an optimization problem with quadratic equality constraints.

A different least squares approach for solving a localization problem with noisy distance measurements was proposed in [12] where *linear* equality constraints were derived. Suppose sensor 0 can measure its distances to beacons 1, 2, ..., r . For beacon i , $i \in \{1, \dots, r\}$, draw a circle, denoted by c_i , centered at beacon i 's position \mathbf{p}_i with radius \bar{d}_{0i} , where \bar{d}_{0i} is the imprecisely measured distance from sensor 0 to beacon i . Then each pair of circles c_i and c_j , $i, j \in \{1, \dots, r\}$ and $i \neq j$, will typically intersect at two points which determine a line. The equation of this line is thus a linear equality constraint used in [12]. When $r > 2$, we can have $r - 1$ such independent linear equality constraints. Although the linear-constraint approach is more computationally efficient than the one we proposed with quadratic constraints, this approach, as we will illustrate by an example in the next section, sometimes fails due to oversimplification during the transformation of intrinsically quadratic constraints into linear ones.

5. A computational example

In this section, we will give an example to demonstrate the steps introduced in previous sections to solve the localization problem. We consider the simplest scenario in 2D as depicted in Fig. 1 where sensor 0, which is the sensor we want to localize, can measure its distances to three beacons 1, 2 and 3 whose coordinates are $\mathbf{p}_1 = (0, 0)$, $\mathbf{p}_2 = (43, 7)$ and $\mathbf{p}_3 = (47, 0)$ respectively. The noisy distance measurements acquired by sensor 0 are $\bar{d}_{01} = 35$, $\bar{d}_{02} = 42$ and $\bar{d}_{03} = 43$.

First, we will determine the quadratic equality constraint as described by Eq. (6). The distances between beacons are

$$d_{12}^2 = d^2(\mathbf{p}_1, \mathbf{p}_2) = 1898,$$

$$d_{13}^2 = d^2(\mathbf{p}_1, \mathbf{p}_3) = 2209,$$

$$d_{23}^2 = d^2(\mathbf{p}_2, \mathbf{p}_3) = 65.$$

Then the matrix A defined by (8) is

$$A = \begin{bmatrix} 130 & -376 & 246 \\ -376 & 4418 & -4042 \\ 246 & 4042 & 3796 \end{bmatrix}.$$

The vector b defined by (9)–(11) is

$$b = [b_1 \ b_2 \ b_3]' \\ = [-623780 \ 805016 \ -1047164]'$$

The scalar c defined by (12) is

$$c = -777892702.$$

Then the quadratic equality constraint for ε_1 , ε_2 and ε_3 defined by (6) is

$$0 = f(\varepsilon_1, \varepsilon_2, \varepsilon_3) \\ = 130\varepsilon_1^2 + 4418\varepsilon_2^2 + 3796\varepsilon_3^2 - 752\varepsilon_1\varepsilon_2 \\ + 492\varepsilon_1\varepsilon_3 - 8084\varepsilon_2\varepsilon_3 - 623780\varepsilon_1 \\ + 805016\varepsilon_2 - 1047164\varepsilon_3 - 777892702.$$

To determine optimal values for ε_1 , ε_2 and ε_3 , we need to solve the following least squares problem:

$$\min \quad \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2, \\ \text{s.t.} \quad f(\varepsilon_1, \varepsilon_2, \varepsilon_3) = 0. \quad (26)$$

We can use the Lagrangian multiplier method to solve this optimization problem. By letting λ be the Lagrangian multiplier, we get the objective function

$$H(\varepsilon_1, \varepsilon_2, \varepsilon_3, \lambda) = \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \lambda f(\varepsilon_1, \varepsilon_2, \varepsilon_3).$$

By differentiating the Lagrangian H with respect to ε_i , $i \in \{1, 2, 3\}$, and letting the result be zero, we have

$$\frac{\partial H}{\partial \varepsilon_1} = 2\varepsilon_1 + \lambda(260\varepsilon_1 - 752\varepsilon_2 + 492\varepsilon_3 - 623780) = 0, \\ \frac{\partial H}{\partial \varepsilon_2} = 2\varepsilon_2 + \lambda(8836\varepsilon_2 - 752\varepsilon_1 - 8084\varepsilon_3 + 805016) = 0, \\ \frac{\partial H}{\partial \varepsilon_3} = 2\varepsilon_3 + \lambda(7592\varepsilon_3 + 492\varepsilon_1 - 8084\varepsilon_2 - 1047164) = 0, \\ \frac{\partial H}{\partial \lambda} = f(\varepsilon_1, \varepsilon_2, \varepsilon_3) = 0.$$

Solving these four algebraic equations numerically and discarding all the nonoptimal stationary-point solutions, the solution for the least squares problem (26) is as follows:

$$\varepsilon_1^* = -37.9590, \quad \varepsilon_2^* = 163.7061 \quad \text{and} \quad \varepsilon_3^* = -164.9748.$$

Correspondingly, the estimated distances from sensor 0 to the three beacons are

$$\hat{d}_{01} = \sqrt{\bar{d}_{01}^2 + \varepsilon_1^*} = 34.4535, \\ \hat{d}_{02} = \sqrt{\bar{d}_{02}^2 + \varepsilon_2^*} = 43.9057, \\ \hat{d}_{03} = \sqrt{\bar{d}_{03}^2 + \varepsilon_3^*} = 41.0369. \quad (27)$$

Now the data in (27) and the positions of beacons 1, 2 and 3 are consistent in the sense that there is a unique point in the plane determined by beacons 1, 2 and 3 whose distances to these three beacons are \hat{d}_{01} , \hat{d}_{02} and \hat{d}_{03} respectively; in other words, we can use triangulation to obtain the estimated sensor location as

$$\hat{\mathbf{p}}_0 = (18.2, -29.2). \quad (28)$$

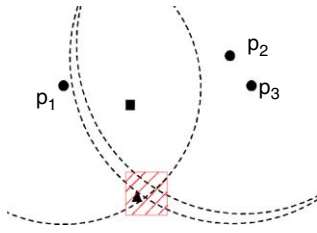


Fig. 3. Locations of beacons 1, 2 and 3, and estimated positions of sensor 0.

As a comparison, we will compute sensor 0's coordinates using the approach formulated in [12] which computes the least squares solution of a set of linear equations. Let (\tilde{x}, \tilde{y}) denote the estimated position of sensor 0 using this approach. For $i \in \{1, 2, 3\}$, the equation for circle c_i centered at beacon i 's position $\mathbf{p}_i = (x_i, y_i)$ with radius \bar{d}_{0i} is

$$(x - x_i)^2 + (y - y_i)^2 = \bar{d}_{0i}^2. \quad (29)$$

By substituting (\tilde{x}, \tilde{y}) into (29) and using the data for \mathbf{p}_i and \bar{d}_{0i} , we have

$$\tilde{x}^2 + \tilde{y}^2 = 35^2, \quad (30)$$

$$(\tilde{x} - 43)^2 + (\tilde{y} - 7)^2 = 42^2, \quad (31)$$

$$(\tilde{x} - 47)^2 + \tilde{y}^2 = 43^2. \quad (32)$$

Subtracting (30) from (31), we can have one linear equality; and subtracting (30) from (32), we obtain another linear equality. These two linear equalities can be written into matrix form

$$H \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = h, \quad (33)$$

where

$$H = \begin{bmatrix} 43 & 7 \\ 47 & 0 \end{bmatrix} \quad \text{and} \quad h = \begin{bmatrix} 679.5 \\ 792.5 \end{bmatrix}.$$

Then the least squares solution [12] of Eq. (33) is

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = (H^T H)^{-1} H^T h = \begin{bmatrix} 16.9 \\ -6.5 \end{bmatrix}. \quad (34)$$

Hence, the estimated sensor location achieved by using linear equality constraints is

$$\tilde{\mathbf{p}}_0 = (16.9, -6.5). \quad (35)$$

We will show the computational results $\hat{\mathbf{p}}_0$ and $\tilde{\mathbf{p}}_0$ in Fig. 3. The solid dots denote the anchors' positions \mathbf{p}_1 , \mathbf{p}_2 and \mathbf{p}_3 , and the dotted circles are the circles c_1 , c_2 and c_3 . The solid triangle denotes the estimated position $\hat{\mathbf{p}}_0$ of sensor 0 in (28) achieved using the approach proposed by this paper. The solid square denotes the estimated position $\tilde{\mathbf{p}}_0$ of sensor 0 in (35) achieved using the approach discussed in [12].

In most real sensor networks, distance measurement errors can be upper and lower bounded by constants, in which case sensor 0's real position lives within a neighborhood of each circle centered at a beacons' position with radius being the measured distance from sensor 0 to the corresponding beacon. This

implies that sensor 0's real position is roughly within the shaded area shown in the figure containing a set of pairwise intersection points of the three circles. Hence, our approach using the quadratic equality constraints derived from the Cayley–Menger determinate gives a better estimate.

6. Conclusions

This paper introduces the Cayley–Menger determinant as an important tool for formulating the geometric relations among node positions in sensor networks as quadratic constraints. It also discusses solutions to optimization problems to estimate the errors in the inaccurate measured distances between sensor nodes and anchor nodes. The solution of the optimization problem, when used to adjust noisy distance measurements, gives a set of distances between nodes which are completely consistent with the fact that sensor nodes live in the same plane or 3D-space as the anchor nodes. These techniques serve as the foundation for most of the existing localization algorithms that depend on the sensors' distances to anchors to compute sensor locations.

For future work, we will apply the technique presented in this paper to the existing localization algorithms and determine the most appropriate objective functions in the optimization process. Other optimization techniques such as semi-definite programming can also be exploited to accelerate the computation processes.

More generally, the calculations presented here should be seen as constituting but one component of a conceptually larger algorithm encompassing the whole localization process in the presence of noisy measurements. A key task is to accommodate uncertainty in the positions of what were regarded in this paper as anchor nodes, since in an actual localization problem, these may be nodes which are not anchor nodes, but simply nodes that were localized at an earlier stage in the overall localization algorithm. Localization of a sequence of vertices would then take into account both uncertainties in the “anchor” nodes and uncertainties in the measured distances, and may derive a nominal position for the just localized node together with an uncertainty region around it. The uncertainty region might be one which contains the newly localized node with a given threshold probability. The way uncertainties propagate will be important; there will be error cumulation that can only be combated by introducing more anchor nodes whose positions are exactly known. A further step would be to incorporate the notion of robust quadrilaterals described in [8].

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