COORDINATION OF AN ASYNCHRONOUS MULTI-AGENT SYSTEM VIA AVERAGING

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Abstract: This paper is concerned with the coordination of a group of $n > 1$ mobile autonomous agents which all move in the plane with the same speed but with different headings. Each agent updates its heading from time to time to a new value equal to the average of its present heading and the headings of its current "neighbors". Although all agents use the same rule, individual updates are executed asynchronously. By appealing to the concept of "analytic synchronization", it is shown that under mild connectivity assumptions of the underlying directed graph characterizing neighbor relationships, the local update rules under consideration can cause all agents to eventually move in the same direction despite the absence of centralized coordination and despite the fact that each agent's set of neighbors change with time as the system evolves.

Keywords: Cooperative control, graph theory, analytic synchronization

1. INTRODUCTION

In recent years, there has been a rapidly growing interest among control scientists and engineers in the distributed coordination of groups of autonomous agents. One of the research directions is to provide theoretical explanations for various collective motions of animal aggregations, such as bird flocking, fish schooling, etc. (Jababaie *et al.*, 2003; Gazi and Passino, 2003; Cortes and Bullo, 2003; Leonard and Fiorelli, 2001). The alignment phenomenon is formulated as a flocking problem (Jababaie *et al.*, 2003), which is later generalized as a network consensus problem (Olfati-Saber and Murray, 2004). This paper studies the flocking problem under the weaker assumption, namely that individual agent's decisions are performed *asynchronously*. In particular, we consider a system consisting of n mobile autonomous agents labelled 1 through n, which are all moving in the plane with the same speed but with different headings. Each agent updates its heading from time to time to a new value equal to the average of its present heading and the headings of its current "neighbors". By a *neighbor* of agent i at time t is meant any other agent whose heading informa-

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tion at time t is available to agent i. Distributed systems of this type have been studied before in (Jababaie *et al.*, 2003; Moreau, 2003) and elsewhere. What distinguishes the system considered here is that in this paper heading updating is done totally asynchronously while in (Jababaie *et al.*, 2003; Moreau, 2003) it is not. In the present context individual agents can update their headings whenever they please; in other words, there is no assumed global clock according to which they synchronize their actions.

We formulate the asynchronous flocking problem in section 2. In section 3 we outline a technique to model a family of asynchronously functioning processes as a single synchronous system. In section 4 we show our main results on proving that the headings of all n agents converge to a common steady state heading provided that they are all "linked together". Finally, concluding remarks are made in section 5.

2. ASYNCHRONOUS FLOCKING PROBLEM

Consider a system that consists of n autonomous agents, labelled 1 through n, which all move in the plane with the same speed but with different headings. Each agent's heading x_i is updated using a simple local rule based on the average of its own heading and the headings of its "neighbors". Agent i's *neighbors* at time t are those agents whose heading information at time t is available to agent i. Agent i's neighbors at any other time do not have to be the same as those at time t. Our results apply to any definition of neighbors, and we do not restrict the definition to constitute a symmetric relation. In other words, agent i is agent j's neighbor at time t does not necessarily imply that agent j is agent i's neighbor at time t. Let $\mathcal{N}_i(t)$ and $n_i(t)$ denote the set of the labels and the number of agent i's neighbors at time t respectively. Let $\{t_{ik} : i \in \{1, \ldots, n\}, k \geq 1\}$ denote the set of agent i's ordered distinct "event times", where by an agent i's *event time* is meant any time t_{ik} at which agent i updates its heading. We assume that each agent updates its heading instantaneously according to the rule

$$x_i(t_{ik}^+) = \frac{1}{1 + n_i(t_{ik})} \left(x_i(t_{ik}) + \sum_{j \in \mathcal{N}_i(t_{ik})} x_j(t_{ik}) \right) \quad (1)$$

where $x_i(t_{ik}^+)$ denotes the value of x_i just after t_{ik}, *i.e.* $x_i(t_{ik}^+) = \lim_{t \searrow t_{ik}} x(t)$. At all times other than agent i's event times, agent i's heading does not change:

$$x_i(t^+) = x_i(t), \qquad \text{when} \quad t_{ik} < t < t_{i(k+1)} \quad (2)$$

So $x_i(t)$ is a piecewise constant function which is constant on each switching interval $[t_{ik}, t_{i(k+1)})$, $k \geq 1$. We assume that the time between such updates is bounded below by a positive number τ_B called a dwell time. In the unlikely event that two or more agents update their headings at the same time T, they all use the values of their respective neighbors' headings at the time immediately prior to T, namely $\lim_{t \nearrow T} x_j(t)$. Event times of distinct agents occur asynchronously, *i.e.* there is no correlation between the event times of different agents.

Our goal is to show that under some connectivity conditions for neighbor relationship graphs, for any initial set of agent headings, the headings of all agents will converge to the same steady state value x_{ss}. There are two main challenges: one is how to analyze an asynchronous distributed system, and the other is how to guarantee the convergence under changing neighbor relationships.

3. A SYNCHRONOUS MODEL

Now we will analyze the asymptotic behavior of the overall asynchronous multi-agent process just described. Despite the apparent complexity of this process, it is possible to capture its salient features using a suitably defined *synchronous* discrete-time system $\mathbb{S}$. We call the sequence of steps involved in defining $\mathbb{S}$ *analytic synchronization* (Lin, 2004). Analytic synchronization is applicable to any finite family of continuous or discrete time dynamical processes $\{\mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_n\}$ under the following conditions. First, each process $\mathbb{P}_i$ must be a dynamical system whose inputs consist of functions of the states of the other processes as well as signals which are exogenous to the entire family. Second, each process $\mathbb{P}_i$ must have associated with it an ordered sequence of event times $\{t_{i1}, t_{i2}, \ldots\}$ defined in such a way so that the state of $\mathbb{P}_i$ at event time $t_{i(k_i+1)}$ is uniquely determined by values of the exogenous signals and states of the $\mathbb{P}_j$, $j \in \{1, 2, \ldots, n\}$ at event times t_{jk_j} which occur prior to $t_{i(k_i+1)}$ but in the uniformly bounded finite past. Event time sequences for different processes need not be synchronized. Analytic synchronization is a procedure for creating a single synchronous process for purposes of analysis which captures the salient features of the original n asynchronously functioning processes. As a first step, all n event time sequences are merged into a single ordered sequence of event times $\mathcal{T}$. The "synchronized" state of $\mathbb{P}_i$ is then defined to be the original state of $\mathbb{P}_i$ at $\mathbb{P}_i$'s event times $\{t_{i1}, t_{i2}, \ldots\}$; at values of $t \in \mathcal{T}$ between event times t_{ik_i} and $t_{i(k_i+1)}$, the synchronized state of $\mathbb{P}_i$ is taken to be the same at the value of its original state at time t_{ik_i}. The last step is to define $\mathbb{S}$ as a synchronous dynamical system evolving on $\mathcal{T}$ with state containing as "substates" the synchronized states of the n individual processes under consideration. Although this last step is in general challenging, in the present paper it proves to be straightforward. We refer readers to (Bertsekas and Tsitsiklis, 1989) for some related ideas.

3.1 Global Time Axis

Let $\mathcal{T} \triangleq \{t_{ik} : i \in \{1, 2, \ldots, n\}, k \geq 1\}$ denote the set of all distinct event times of all n agents. Relabel this set's elements as $t_0, t_1, \ldots, t_l, \ldots$ in such a way so that $t_l < t_{l+1}$, $l \in \{0, 1, 2, \ldots\}$. Define $\mathcal{T}_i$ to be the ordered subset of $\mathcal{T}$ consisting of agent i's event times. For $i \in \{1, 2, \ldots, n\}$, let $L_i(k)$ denote that value of l for which $t_l = t_{ik}$. Thus with this notation, $t_{L_i(k)} = t_{ik}$. Then each agent's heading is also well defined at any other agent's event times according to its evolution equations (1) and (2).

The evolution of agent i's heading on the set $\mathcal{T}$ can thus be described by

$$x_i(t_{l+1}) = \begin{cases} \dfrac{1}{1+n_i(t_l)} \left(x_i(t_l) + \displaystyle\sum_{j \in \mathcal{N}_i(t_l)} x_j(t_l) \right) & \\ \qquad\qquad \text{if } t_l \in \mathcal{T}_i & \\ x_i(t_l) \qquad \text{otherwise} & \end{cases} \quad (3)$$

3.2 System Equations and Underlying Directed Graphs

The update equations determined by (3) depend on the neighbor relationships which exist at time t_l. We define a directed graph with vertex set $\{1, \ldots, n\}$ so that for $j \neq i$, (j, i) is a directed edge from vertex j to vertex i if and only if $t_l \in \mathcal{T}_i$ and agent j is a neighbor of agent i at time t_l. To account for the fact that agent i's own heading information is always used in its heading update rule, we also say that in the just defined directed graph, there is always a directed edge from agent i to itself. At different event times, we have different directed graphs just described because different agents update their headings spontaneously and the set of neighbors of any agent, for example i, is not necessarily constant over agent i's event times. To account for this we will need to consider all possible such graphs, which are obviously finite. In the sequel we use the symbol $\mathcal{P}$ to denote a suitably defined set indexing the class of all directed graphs $\mathbb{G}_p$ defined on n vertices.

We can define the system state as

$$x = [x_1, \ldots, x_n]' \quad (4)$$

In order to write the system equations in state-space form, for each $p \in \mathcal{P}$, we define

$$F_p = (D_p)^{-1} A_p \quad (5)$$

where A_p is the "adjacency matrix" of graph $\mathbb{G}_p$ and D_p is the diagonal matrix whose ith diagonal element is the "valence" of vertex i within the graph $\mathbb{G}_p$. The *adjacency matrix* $A(\mathbb{G})$ of a directed graph $\mathbb{G}$ is a 0-1 matrix with the rows and columns indexed by the vertices of $\mathbb{G}$, such that the ij-element of $A(\mathbb{G})$ is equal to 1 if there is a directed edge from vertex j to i and is equal to 0 otherwise. The *valence* of a vertex i is the number of directed edges that end at vertex i.

Because the global system now can be viewed as a discrete synchronous system, the system evolution can be written as the following equations

$$x(t_{l+1}) = F_{\sigma(t_l)} x(t_l), \qquad t_l \in \mathcal{T} \quad (6)$$

where $\sigma : \mathcal{T} \to \mathcal{P}$ is a switching signal whose value at time t_l is the index of the directed graph representing the agents' neighbor relationships at time t_l. From (3), we know that $F_{\sigma(t_l)}$ has positive diagonal elements. Notice that at time $t_l \notin \mathcal{T}_i$, there is no edge pointing from the other vertices towards vertex i in $\mathbb{G}_{\sigma(t_l)}$ and consequently the elements of the ith row in $F_{\sigma(t_l)}$ are $(F_{\sigma(t_l)})_{ii} = 1$ and $(F_{\sigma(t_l)})_{ij} = 0$ with $j \neq i$ and $j = 1, \ldots, n$.

3.3 Connectivity of the Underlying Directed Graphs

A *strongly connected* directed graph is a directed graph in which it is possible to reach any vertex starting from any other vertex by traversing a sequence of edges in the direction to which they point. A *weakly connected* directed graph is a directed graph in which it is possible to reach any vertex starting from any other vertex by traversing edges in some direction (*i.e.* not necessarily in the direction they point). A *global source* (often simply called a *source*) in a directed graph is a vertex which can reach all other vertices by traversing a sequence of edges in the direction to which they point. As we can see, a directed graph having a source is weakly connected while the reverse is not always true.

Consider two directed graphs, $\mathbb{G}_{p1}$ and $\mathbb{G}_{p2}$, each with vertex set $\mathcal{V}$. Let $\mathbb{G}_{p2} \circ \mathbb{G}_{p1}$ denote the *composition* of $\mathbb{G}_{p1}$ and $\mathbb{G}_{p2}$, which is defined as the directed graph $\mathbb{G}$ with the vertex set $\mathcal{V}$ and the following property: there is a directed edge from vertex i to j within $\mathbb{G}$ if and only if there exists a vertex $k \in \mathcal{V}$ such that directed edges ik and kj are within $\mathbb{G}_{p1}$ and $\mathbb{G}_{p2}$ respectively. In general, this operation on the graphs is not commutative, *i.e.* $\mathbb{G}_{p2} \circ \mathbb{G}_{p1} \neq \mathbb{G}_{p1} \circ \mathbb{G}_{p2}$. Across a time interval $[t, \tau)$ with $t, \tau \in \mathcal{T}$ and $\tau > t$, the n agents under consideration are said to be *linked together* if the composition of graphs, $\mathbb{G}_{\sigma(\tau-1)} \circ \cdots \circ \mathbb{G}_{\sigma(t+1)} \circ \mathbb{G}_{\sigma(t)}$, encountered along the interval has at least one source. Notice that if n agents are linked together and there is only one source in the composition of the graphs, then all agents except the source agent must have updated at least once across $[t, \tau)$; if n agents are linked together and there is more than one source in the composition of the graphs, then all n agents must have updated at least once across $[t, \tau)$.

4. MAIN RESULTS

Reference (Jababaie *et al.*, 2003) provides a convergence result by analyzing the product of the matrices associated with underlying graphs, whose entries are all positive under the assumption that the corresponding underlying graphs are strongly connected. In the sequel, we will consider a weaker condition for system (6) when its composition of underlying graphs is only weakly connected. So the bidirectional information flow between any pair of agents is no longer guaranteed. Consequently, some entries of the product of the corresponding matrices might always be zero, and we cannot obtain the element-wise positiveness that was achieved in (Jababaie *et al.*, 2003). Another distinction is that by considering the composition of the directed graphs, we take into account the time ordering information of the directed graphs, which is not considered in (Jababaie *et al.*, 2003).

4.1 Weakly Connected Graphs

We first consider a special case. Intuitively, the flocking phenomenon is highly likely to happen if there is a particular agent whose heading information is always available to all the other agents during the evolution process. Now we will give a strict proof of convergence in this case. In the set of directed graphs $\{\mathbb{G}_p : p \in \mathcal{P}\}$, we denote by $\mathcal{Q}$ the subset of $\mathcal{P}$ with the property that there exists a vertex s common to all $q \in \mathcal{Q}$ such that there is a directed edge from s to each of the other vertices in $\mathbb{G}_q$. Thus agent s is a neighbor of every other agent at every event time. Our first result establishes the convergence of x for the case when σ takes values only in $\mathcal{Q}$.

Theorem 1. Let $x(t_0)$ be fixed and let $\sigma : \mathcal{T} \to \mathcal{P}$ be a switching signal satisfying $\sigma(t) \in \mathcal{Q}$ for all $t_l \in \mathcal{T}$, *i.e.* there exists a particular agent s that is a neighbor of all other agents at all times $t_l \in \mathcal{T}$. Then

$$\lim_{t\to\infty} x(t) = x_{ss}\mathbf{1} \tag{7}$$

where x_{ss} is a number depending only on $x(t_0)$ and σ, $\mathbf{1} \triangleq [1 \quad 1 \quad \ldots \quad 1]'_{n\times 1}$.

To prove Theorem 1, we need to introduce some related definitions and convergence results in matrix analysis.

As defined by (5), F_p is square and "non-negative" (a *non-negative matrix* is a matrix whose elements are all non-negative) with the property that its row sums all equal 1 (*i.e.* $F_p\mathbf{1} = \mathbf{1}$). Such a matrix is called *stochastic* (Horn and Johnson, 1985). F_p has the additional property that its diagonal elements are all positive. For the case when $p \in \mathcal{Q}$, F_p is an example of "scrambling matrices" where by a *scrambling matrix* is meant any n-by-n, non-negative matrix M with the property that for arbitrary i and j with $i, j \in \{1, \ldots, n\}$, $i \neq j$, there exists $k \in \{1, \ldots, n\}$ such that m_{ik} and m_{jk} are both positive (Shen, 2000).

Lemma 2. F_p is a scrambling matrix when $p \in \mathcal{Q}$.

PROOF. Because vertex s is the source and there is an directed edge from s to each of the other vertices, we have $(F_p)_{rs} > 0$, $r \in \{1, 2, \ldots, n\}$ and $r \neq s$. From the fact that F_p has positive diagonal elements, we also have $(F_p)_{ss} > 0$. Then for arbitrary i and j, $i, j \in \{1, \ldots, n\}$, $i \neq j$, we always have $(F_p)_{is} > 0$ and $(F_p)_{js} > 0$, hence F_p is scrambling. □

Now we will consider the convergence of the product of scrambling matrices. First we point out that the class of n-by-n stochastic matrices with positive diagonal elements is closed under matrix multiplication. Second, we will re-phrase Proposition 27 and its proof in (Shen, 2000) as follows:

Lemma 3. (Proposition 27 in (Shen, 2000)) Let $\mathcal{M} = \{M_i\}, i = 1, 2, \ldots$ be a compact set of scrambling stochastic matrices. Then for each infinite sequence $M_{i_1}, M_{i_2}, \ldots$ there exists a row vector c such that

$$\lim_{j\to\infty} M_{i_j} M_{i_{j-1}} \cdots M_{i_1} = \mathbf{1}c \tag{8}$$

PROOF. We first point out the fact that for a given constant row vector b and any stochastic matrix A, we have $A(\mathbf{1}b) = \mathbf{1}b$ because of the fact that $A\mathbf{1} = \mathbf{1}$. For any $M \in \mathcal{M}$, define $\lambda(M) = \min_{i,j} \sum_{k=1}^{n} \min\{m_{ik}, m_{jk}\}$. Then $\lambda(M) > 0$ because M is scrambling. Because M is stochastic, $\lambda(M) \leq 1$ with $\lambda(M) = 1$ if and only if all the rows of matrix M are the same. Since $\mathcal{M}$ is compact, we can define

$$\lambda(\mathcal{M}) \triangleq \inf_{M\in\mathcal{M}} \lambda(M) > 0$$

Denote the ith row of matrix M by m'_i. Then define

$$\text{diam}_\infty(M) \triangleq \max_{i,j} \|m_i - m_j\|_\infty$$

where for a vector $a = [a_1\ a_2\ \dots\ a_n]'$, $\|a\|_\infty = \max_i\ |a_i|$. Then Hajnal's inequality (Shen, 2000) says that for any stochastic matrices A_1 and A_2, we have $\text{diam}_\infty(A_2A_1) \leq (1-\lambda(A_2))\text{diam}_\infty(A_1)$, hence

$$\text{diam}_\infty(M_{i_j}M_{i_{j-1}}\cdots M_{i_1}) \leq (1-\lambda(\mathcal{M}))^j \text{diam}_\infty(I)$$

where I is the n-by-n identity matrix. Because $0 < \lambda(\mathcal{M}) \leq 1$, then as $j \to \infty$, we have $\text{diam}_\infty(M_{i_j}M_{i_{j-1}}\cdots M_{i_1}) \to 0$, which implies that the element-wise difference between any pair of rows in the product $M_{i_j}M_{i_{j-1}}\cdots M_{i_1}$ approaches 0. Then (8) holds. □

Now we are in a position to prove Theorem 1.

Proof of Theorem 1: From (6), we have

$$x(t_j) = F_{\sigma(t_{j-1})}\cdots F_{\sigma(t_0)}x(t_0)$$

By Lemma 2, for $\sigma(t_j) \in \mathcal{Q}$, the set of possible $F_{\sigma(t_j)}$, $j \geq 0$, is a finite set of scrambling matrices. Hence, by Lemma 3 there exists a row vector c such that

$$\lim_{j\to\infty} F_{\sigma(t_j)}F_{\sigma(t_{j-1})}\cdots F_{\sigma(t_0)} = \mathbf{1}c$$

Letting $x_{ss} \triangleq cx(t_0)$, we arrive at (7). □

4.2 Generalization

It is possible to establish convergence to a common heading under conditions which are significantly less stringent than those assumed in Theorem 1. We now consider a situation where the agents are linked together across some finite-length intervals. In this case, the convergence of the n-agent system is still guaranteed while some pairs of agents may never be strongly connected.

Given a sequence of finite-length time intervals across which all n agents are linked together, our first result claims that there exists a finite-length time interval such that the composition of the directed graphs encountered along this interval is an element in the set $\{\mathbb{G}_p: p \in \mathcal{Q}\}$.

Theorem 4. Given any starting time $t_{i_0} \in \mathcal{T}$, assume that there exists an infinite sequence of contiguous, non-empty, bounded time-intervals $[t_{i_j}, t_{i_{j+1}})$, $t_{i_j} \in \mathcal{T}$, $j \geq 0$, starting at t_{i_0}, with the property that across each such interval, the n agents are linked together with one agent s as the source in every such interval. Then there exists a non-empty bounded time-interval $[t_{i_0}, t_{i_k})$, $t_{i_k} \in \mathcal{T}$, $k > 0$, such that across $[t_{i_0}, t_{i_k})$, there is a directed edge from vertex s to every other vertex in the composition of the corresponding graphs.

To prove this theorem, we need the following definition and results. We shall make use of the standard partial ordering $\geq$ on $n\times n$ non-negative matrices by writing $B \geq A$ whenever $B - A$ is a non-negative matrix. For a non-negative matrix R, we denote by $\lceil R \rceil$, the matrix obtained by replacing all of R's non-zero entries with 1s. Notice that R is scrambling if and only if $\lceil R \rceil$ is scrambling. It is true that for any pair of $n\times n$ non-negative matrices A and B with positive diagonal elements, that $\lceil AB \rceil = \lceil \lceil A \rceil \lceil B \rceil \rceil$, $\lceil AB \rceil \geq \lceil B \rceil$ and $\lceil BA \rceil \geq \lceil A \rceil$. The following lemma is Lemma 6 in (Jababaie *et al.*, 2003).

Lemma 5. (Lemma 6 in (Jababaie *et al.*, 2003)) Let $M_1, M_2, \dots, M_k$ be a finite sequence of n-by-n non-negative matrices whose diagonal entries are all positive. Suppose that M is a matrix which occurs in the sequence at least $m > 0$ times. Then

$$\lceil M_1M_2\cdots M_k \rceil \geq \lceil M^m \rceil \quad (9)$$

Proof of Theorem 4: Let H_{t_j} associated with the time interval $[t_{i_j}, t_{i_{j+1}})$ be the product of matrices as in $H_{t_j} = \lceil F_{\sigma(t_{i_{j+1}-1})}\cdots F_{\sigma(t_{i_j}+1)}F_{\sigma(t_{i_j})} \rceil$. Then H_{t_j} is also associated with the composition of the directed graphs encountered along this time interval with the property that s is a source in this composition. There are only a finite number of all possible H_{t_j}. Let $\mathcal{H}$ denote the set of all possible H_{t_j} and let a finite number w denote the number of elements in $\mathcal{H}$. Then we can relabel $\mathcal{H}$'s elements as H_1, H_2,…,H_w. It's easy to see that for some $1 \leq u \leq w$, there exists a finite length interval $[t_{i_0}, t_{i_k})$ such that H_u occurs at least n times in the corresponding sequence of matrices, $H_{t_{i_0}}$, $H_{t_{i_1}}$,…, $H_{t_{i_{k-1}}}$. From Lemma 5, we know that

$$\lceil H_{t_{i_{k-1}}}\cdots H_{t_{i_1}}H_{t_{i_0}} \rceil \geq \lceil (H_u)^n \rceil \quad (10)$$

Suppose that within the composition of graphs associated with H_u, vertex s can reach vertex r ($r \in \{1,\dots,n\}$ and $r \neq s$) by traversing q edges in the direction in which the edges point, where obviously $0 < q < n$. Then $((H_u)^q)_{sr} > 0$. Since there is always a directed edge from vertex i to itself, if s can reach r in q steps, then s can also reach r in n steps. This implies $((H_u)^n)_{sr} > 0$. Hence, from (10) we know that across $[t_{i_0}, t_{i_k})$, there is a directed edge from vertex s to any other vertex within the composition of the corresponding graphs. □

Now we are in a position to present the convergence result.

Theorem 6. Let $x(t_0)$ be fixed and let $\sigma : \mathcal{T} \to \mathcal{P}$ be a switching signal for which there exists an infinite sequence of contiguous, non-empty, bounded time-intervals $[t_{i_j}, t_{i_{j+1}})$, $t_{i_j} \in \mathcal{T}$, $j \geq 0$, starting at $t_{i_0} = t_0$, with the property that across each such interval, the n agents are linked together with agent s as the source. Then

$$\lim_{t\to\infty} x(t) = x_{ss}\mathbf{1} \tag{11}$$

where x_{ss} is a number depending only on $x(t_0)$ and σ.

PROOF. From Theorem 4, with the given switching signal, there exists a sequence of contiguous, non-empty, bounded time-intervals $[t_{j_k}, t_{j_{k+1}})$, with $t_{j_k} \in \mathcal{T}$, $t_{j_0} = t_0$, such that agent s is a neighbor of every other agent at time $t_{j_{k+1}}$ in the composition of graphs encountered along $[t_{j_k}, t_{j_{k+1}})$. Similar to the proof of Theorem 1, we can show that the matrix $F_{\sigma(t_{j_{k+1}}-1)}F_{\sigma(t_{j_{k+1}}-2)}\cdots F_{\sigma(t_{j_k})}$ is a scrambling matrix. Then

$$\begin{aligned} x(t_{j_k}) &= F_{\sigma(t_{j_k}-1)}\cdots F_{\sigma(t_0)}x(t_0) \\ &= \Big(F_{\sigma(t_{j_k}-1)}\cdots F_{\sigma(t_{j_{k-1}})}\Big) \\ &\quad \cdots \Big(F_{\sigma(t_{j_2}-1)}\cdots F_{\sigma(t_{j_1})}\Big) \\ &\quad \Big(F_{\sigma(t_{j_1}-1)}\cdots F_{\sigma(t_0)}\Big)\, x(t_0) \end{aligned}$$

By a similar argument as that is used in proving Theorem 1, we can show that there exists a row vector c such that

$$\lim_{k\to\infty} F_{\sigma(t_{j_k})}\cdots F_{\sigma(t_0)} = \mathbf{1}c$$

Letting $x_{ss} \triangleq cx(t_0)$, we arrive at (11). □

5. CONCLUDING REMARKS

This paper provides an approach to studying the asynchronous flocking problem. A modelling technique is first introduced to construct a synchronous model for analysis purposes. Then a relationship is established between a class of weakly connected graphs and scrambling matrices. The convergence property of the product of scrambling matrices is utilized to prove the convergence of the asynchronous flocking problem.

The asynchronous model is more realistic than existing synchronized multi-agent system models. Consequently, it is more appropriate to use the asynchronous model in the study of actual coordinated movements of groups of mobile robots. In fact, it is not only heading variables that can be averaged during the asynchronous multi-agent processes; we can also consider any physical variable which obeys some update rules like (1) and (2). We can even consider a vector variable, whose entries are updated independently. The heading information in 3-dimensional space is an example of such a vector variable, where each entry will converge at the same rate as determined by the matrices $F_{\sigma(t_j)}$ if neighbor relations are the same across all entries of the heading vector. In the future, we will further consider the communication delays, errors, etc. that may be present in asynchronous functioning processes.

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