

# Localization with Imprecise Distance Information in Sensor Networks

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**Abstract**—An approach to formulate geometric relations among distances between nodes as equality constraints is introduced in this paper to study the localization problem with imprecise distance information in sensor networks. These constraints can be further used to formulate optimization problems for distance estimation. The optimization solutions correspond to a set of distances that are consistent with the fact that sensor nodes live in the same plane or 3D space as the anchor nodes. These techniques serve as the foundation for most of the existing localization algorithms that depend on the sensors' distances to anchors to compute each sensor's location.

## I. INTRODUCTION

In sensor networks, sensors' location information is vital in location-aware applications and influences network performances when algorithms like geographic routing are used. Hence, localization is crucial in the development of low-cost sensor networks where it is not feasible for all the nodes' locations to be directly measurable via GPS or other similar means. The locations of some of the nodes have to be inferred by utilizing estimated distances to their nearby nodes. Hence, as pointed out in [1], the first and the most fundamental phase of most localization algorithms [2], [3], [4] is the determination of the distances between sensor nodes whose locations are to be computed and "anchor nodes". An *anchor node* is a node whose location is assumed to be known during the current computation. An anchor node may be a node with a GPS device, or a node with a tentative estimated location in an iterative computation process [3], or a point in the trajectory

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of a mobile beacon [5], etc. Distances between sensors and anchors can be obtained via direct measurements if they are within sensing range of one another; otherwise, approximation methods such as sum-dist [1], [3] and DV-hop [2], [4] can be used to estimate the sensor-anchor distances. No matter which method is used to obtain the distances, the data acquired are usually imprecise compared with the true distances because of measurement noise and estimation errors. Because the true distances between nodes are interdependent, these inaccuracies have undesirable consequences of causing inconsistency with respect to geometric relations, and sometimes may even cause localization algorithms to collapse. For example, in a 2D scenario, triangulation fails due to nonexistence of feasible solutions when the distances are not consistent with the fact that all sensors live on a plane.

However, these imprecise distances can be made more accurate and consistent by exploiting fully the geometric and algebraic relations among the distances between nodes. One possible approach is to first find a feasible set of consistent distances in the sense that the corresponding sensors' locations live in a given space; then, distances are selected from this feasible set by optimizing a desired objective function such as least squares. Hence, given imprecise distances, the goal is to find the set of distances that minimizes the desired objective function with respect to the given distances, that is also consistent with the requirement that sensors lie in the given space. Using these processed distances the positions of the corresponding sensor nodes can then be determined by lateration [2], [4], min-max optimization [3], or any of the other appropriate methods.

In this paper a novel technique is presented which describes the geometric relations among the sensor-anchor distances as one or multiple quadratic equality constraints. The key step is to use the *Cayley-Menger*

*determinant* that is defined in the following section. Although inequality constraints, usually variations of the triangle inequality where the sum of the lengths of two sides of the triangle must be greater than the third, have been discussed before [6], the equality constraints reported in this paper disclose more insightfully the dependency among distances between nodes. Furthermore, these equality constraints can be utilized to estimate errors in the measured or computed distances. One specific estimation approach presented in this paper is to solve a least squares problem where the objective is to minimize the sum of the squared errors in the distances that are measured or computed by a sensor. In [7], geometric conditions about the locations of the nodes are investigated to guarantee the uniqueness of the localization solution in 2D when imprecise distances are given; the results there can also be integrated into our method.

The rest of the paper is organized as follows. In section 2, the definition of the Cayley-Menger determinant is introduced as well as related classic results in Distance Geometry. In section 3, geometric relations among nodes' positions are formulated as quadratic constraints by using the Cayley-Menger determinant. In section 4, we show that errors in the imprecise distance information can be estimated by solving an optimization problem. Finally, concluding remarks are made in section 5.

## II. CAYLEY-MENGER DETERMINANTS

The *Cayley-Menger Matrix* of two sequences of  $n$  points,  $\{\mathbf{p}_1, \dots, \mathbf{p}_n\}$  and  $\{\mathbf{q}_1, \dots, \mathbf{q}_n\} \in \mathbb{R}^m$ , is defined as

$$M(\mathbf{p}_1, \dots, \mathbf{p}_n; \mathbf{q}_1, \dots, \mathbf{q}_n) \triangleq \begin{bmatrix} d^2(\mathbf{p}_1, \mathbf{q}_1) & d^2(\mathbf{p}_1, \mathbf{q}_2) & \cdots & d^2(\mathbf{p}_1, \mathbf{q}_n) & 1 \\ d^2(\mathbf{p}_2, \mathbf{q}_1) & d^2(\mathbf{p}_2, \mathbf{q}_2) & \cdots & d^2(\mathbf{p}_2, \mathbf{q}_n) & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d^2(\mathbf{p}_n, \mathbf{q}_1) & d^2(\mathbf{p}_n, \mathbf{q}_2) & \cdots & d^2(\mathbf{p}_n, \mathbf{q}_n) & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix} \quad (1)$$

where  $d(\mathbf{p}_i, \mathbf{q}_j)$ ,  $i, j \in \{1, \dots, n\}$  is the Euclidean distance between the points  $\mathbf{p}_i$  and  $\mathbf{q}_j$ . The *Cayley-Menger bideterminant* [8] of these two sequences of  $n$  points is defined as

$$D(\mathbf{p}_1, \dots, \mathbf{p}_n; \mathbf{q}_1, \dots, \mathbf{q}_n) \triangleq \det M(\mathbf{p}_1, \dots, \mathbf{p}_n; \mathbf{q}_1, \dots, \mathbf{q}_n) \quad (2)$$

This determinant is widely used in Distance Geometry theory [9], [8] which deals with Euclidean geome-

try in spaces where "distance" is defined and invariant. When the two sequences of points are the same,  $M(\mathbf{p}_1, \dots, \mathbf{p}_n; \mathbf{p}_1, \dots, \mathbf{p}_n)$  and  $D(\mathbf{p}_1, \dots, \mathbf{p}_n; \mathbf{p}_1, \dots, \mathbf{p}_n)$  are denoted for convenience by  $M(\mathbf{p}_1, \dots, \mathbf{p}_n)$  and  $D(\mathbf{p}_1, \dots, \mathbf{p}_n)$  respectively, and the latter is simply called a *Cayley-Menger determinant*.

The Cayley-Menger determinant provides another way of expressing the hyper-volume of a "simplex" by using only the lengths of the edges. A *simplex* of  $n$  points is the smallest  $(n-1)$ -dimensional convex hull containing these points. The hyper-volume  $V$  of the simplex formed by the points  $\mathbf{p}_1, \dots, \mathbf{p}_n$  is given by

$$V^2(\mathbf{p}_1, \dots, \mathbf{p}_n) = \frac{(-1)^n}{2^{n-1}((n-1)!)^2} D(\mathbf{p}_1, \dots, \mathbf{p}_n) \quad (3)$$

We can check equation (3) for the following low dimensional cases:

- For  $n=2$ ,  $D(\mathbf{p}_1, \mathbf{p}_2) = 2d^2(\mathbf{p}_1, \mathbf{p}_2)$ , and  $V(\mathbf{p}_1, \mathbf{p}_2) = d(\mathbf{p}_1, \mathbf{p}_2)$ .
- For  $n=3$ , the simplex is the triangle formed by  $\mathbf{p}_1$ ,  $\mathbf{p}_2$ , and  $\mathbf{p}_3$ . Then  $V(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$  is the area of this triangle. Let  $a, b, c$  be the lengths of the three edges of the triangle, namely  $a = d(\mathbf{p}_1, \mathbf{p}_2)$ ,  $b = d(\mathbf{p}_2, \mathbf{p}_3)$ ,  $c = d(\mathbf{p}_3, \mathbf{p}_1)$ . Let  $s$  denote the semi-perimeter  $s = \frac{1}{2}(a+b+c)$ . Then from Heron's formula [10], we know that  $V(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = \sqrt{s(s-a)(s-b)(s-c)}$ . Hence, it is easy to check that  $V^2(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = \frac{-1}{16} D(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ .
- For  $n=4$ , the simplex is the tetrahedron formed by  $\mathbf{p}_1$ ,  $\mathbf{p}_2$ ,  $\mathbf{p}_3$ , and  $\mathbf{p}_4$ . We can obtain Euler's formula [11] relating the volume of a tetrahedron with its edge-lengths:  $V^2(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = \frac{1}{288} D(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4)$ .

The following theorem is a classical result on the Cayley-Menger determinant and is later generalized in [12].

*Theorem 1:* Consider an  $n$ -tuple of points  $\mathbf{p}_1, \dots, \mathbf{p}_n$  in  $m$ -dimensional space. If  $n \geq m+2$ , then the Cayley-Menger matrix  $M(\mathbf{p}_1, \dots, \mathbf{p}_n)$  is singular, namely

$$D(\mathbf{p}_1, \dots, \mathbf{p}_n) = 0 \quad (4)$$

A stronger statement can be made as follows in terms of the rank of the Cayley-Menger matrix.

*Theorem 2:* (Theorem 112.1 in [9]) Consider an  $n$ -tuple of points  $\mathbf{p}_1, \dots, \mathbf{p}_n$  in  $m$ -dimensional space with  $n \geq m+1$ . The rank of the Cayley-Menger matrix  $M(\mathbf{p}_1, \dots, \mathbf{p}_n)$  is at most  $m+1$ .

In fact, the rank of  $M(\mathbf{p}_1, \dots, \mathbf{p}_n)$  equals  $m+1$  if and

only if at least  $m + 1$  points of the  $n$  points are in generic positions. A similar statement made in terms of the cofactors of the Cayley-Menger determinant can be found in Corollary 1 of [12].

### III. GEOMETRIC RELATIONS AS EQUALITY CONSTRAINTS

In this section, we will illustrate how one can describe the geometric relations among the distances between nodes as algebraic constraints, which are, to be precise, quadratic algebraic equations. We first consider the two dimensional case, and the result will then be generalized to the three dimensional case.

#### A. 2D Case

As shown in Fig. 1, nodes 1, 2 and 3 are anchor nodes and node 0 is a sensor node with unknown position to be localized. We assume that nodes 1, 2 and 3 are in non-collinear positions.

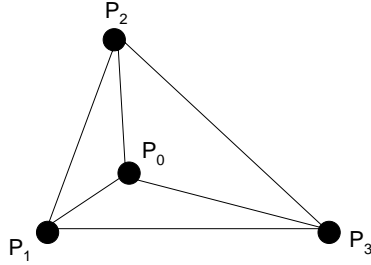


Fig. 1 Sensor node 0 with anchor nodes 1, 2 and 3 in 2D

Let  $d_{ij} = d(\mathbf{p}_i, \mathbf{p}_j)$  denote the accurate Euclidean distance between nodes  $i$  and  $j$  with  $i, j = 0, 1, 2, 3$ . Now suppose that inaccurate distances  $\bar{d}_{0i}$ ,  $i = 1, 2, 3$  acquired by either noisy range measurements or computations [2], [3], [4], are available while the  $d_{ij}$ , the accurate Euclidean distances between anchor nodes with  $i \neq j$ ,  $i, j = 1, 2, 3$ , are known. Then

$$\bar{d}_{0i}^2 = d_{0i}^2 - \varepsilon_i \quad (5)$$

for some  $\varepsilon_i$ .

*Theorem 3:* The errors  $\varepsilon_i$  for  $i = 1, 2, 3$  as defined immediately above satisfy a single algebraic equality which is quadratic though not homogeneous in the  $\varepsilon_i$ 's, and whose coefficients are determined by  $\bar{d}_{0i}$  for  $i = 1, 2, 3$  and  $d_{ij}$  for  $i, j = 1, 2, 3$  and  $i \neq j$ :

$$\varepsilon^T A \varepsilon + \varepsilon^T b + c = 0 \quad (6)$$

where

$$\varepsilon = [\varepsilon_1 \quad \varepsilon_2 \quad \varepsilon_3]^T \quad (7)$$

$$A = \begin{bmatrix} 2d_{23}^2 & d_{12}^2 - d_{13}^2 - d_{23}^2 & d_{13}^2 - d_{23}^2 - d_{12}^2 \\ d_{12}^2 - d_{13}^2 - d_{23}^2 & 2d_{13}^2 & d_{23}^2 - d_{12}^2 - d_{13}^2 \\ d_{13}^2 - d_{12}^2 - d_{23}^2 & d_{23}^2 - d_{12}^2 - d_{13}^2 & 2d_{12}^2 \end{bmatrix} \quad (8)$$

$$b_1 = 4d_{23}^2 \bar{d}_{01}^2 + 2(d_{12}^2 - d_{13}^2 - d_{23}^2) \bar{d}_{02}^2 + 2(d_{13}^2 - d_{12}^2 - d_{23}^2) \bar{d}_{03}^2 + 2d_{23}^2 (d_{23}^2 - d_{12}^2 - d_{13}^2) \quad (9)$$

$$b_2 = 4d_{13}^2 \bar{d}_{02}^2 + 2(d_{12}^2 - d_{13}^2 - d_{23}^2) \bar{d}_{01}^2 + 2(d_{23}^2 - d_{12}^2 - d_{13}^2) \bar{d}_{03}^2 + 2d_{13}^2 (d_{13}^2 - d_{12}^2 - d_{23}^2) \quad (10)$$

$$b_3 = 4d_{12}^2 \bar{d}_{03}^2 + 2(d_{13}^2 - d_{12}^2 - d_{23}^2) \bar{d}_{01}^2 + 2(d_{23}^2 - d_{12}^2 - d_{13}^2) \bar{d}_{02}^2 + 2d_{12}^2 (d_{12}^2 - d_{13}^2 - d_{23}^2) \quad (11)$$

$$c = 2d_{12}^2 d_{13}^2 d_{23}^2 + 2d_{23}^2 \bar{d}_{01}^4 + 2d_{13}^2 \bar{d}_{02}^4 + 2d_{12}^2 \bar{d}_{03}^4 + 2(d_{12}^2 - d_{13}^2 - d_{23}^2) \bar{d}_{01}^2 \bar{d}_{02}^2 + 2(d_{13}^2 - d_{12}^2 - d_{23}^2) \bar{d}_{01}^2 \bar{d}_{03}^2 + 2(d_{23}^2 - d_{12}^2 - d_{13}^2) \bar{d}_{02}^2 \bar{d}_{03}^2 + 2d_{23}^2 (d_{23}^2 - d_{12}^2 - d_{13}^2) \bar{d}_{01}^2 + 2d_{13}^2 (d_{13}^2 - d_{12}^2 - d_{23}^2) \bar{d}_{02}^2 + 2d_{12}^2 (d_{12}^2 - d_{13}^2 - d_{23}^2) \bar{d}_{03}^2 \quad (12)$$

*Proof:* From Theorem 1 we know that  $D(\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = 0$ , namely

$$\det \begin{bmatrix} 0 & d_{01}^2 & d_{02}^2 & d_{03}^2 & 1 \\ d_{01}^2 & 0 & d_{12}^2 & d_{13}^2 & 1 \\ d_{02}^2 & d_{12}^2 & 0 & d_{23}^2 & 1 \\ d_{03}^2 & d_{13}^2 & d_{23}^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} = 0 \quad (13)$$

Since nodes 1, 2 and 3 are non-collinear,  $D(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \neq 0$ . So we can define

$$E \triangleq \begin{bmatrix} 0 & d_{12}^2 & d_{13}^2 & 1 \\ d_{12}^2 & 0 & d_{23}^2 & 1 \\ d_{13}^2 & d_{23}^2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}^{-1} \quad (14)$$

Then from (13) we know that

$$\begin{bmatrix} d_{01}^2 & d_{02}^2 & d_{03}^2 & 1 \end{bmatrix} E \begin{bmatrix} d_{01}^2 \\ d_{02}^2 \\ d_{03}^2 \\ 1 \end{bmatrix} = 0 \quad (15)$$

Here we have used the fact that for an arbitrary matrix  $B = \begin{bmatrix} b_{11} & \mathbf{b}_{12} \\ \mathbf{b}_{12}^T & B_{22} \end{bmatrix}$ , where  $b_{11}$  is a scalar, if  $B_{22}$  is non-singular, then

$$\det B = (b_{11} - \mathbf{b}_{12} B_{22}^{-1} \mathbf{b}_{12}^T) \det B_{22}$$

It follows that

$$\begin{bmatrix} \bar{d}_{01}^2 + \varepsilon_1 & \bar{d}_{02}^2 + \varepsilon_2 & \bar{d}_{03}^2 + \varepsilon_3 & 1 \end{bmatrix} E \begin{bmatrix} \bar{d}_{01}^2 + \varepsilon_1 \\ \bar{d}_{02}^2 + \varepsilon_2 \\ \bar{d}_{03}^2 + \varepsilon_3 \\ 1 \end{bmatrix} = 0 \quad (16)$$

which defines a quadratic surface in the  $\varepsilon_i$ 's. Multiplying both sides of (16) by the determinant of the inverse of  $E$ , we arrive at (6). ■

Now we will show the semi-definiteness of matrix  $A$  that is defined in (8).

*Theorem 4:* With the same hypothesis as for Theorem 3 and with  $A$  as defined in (8), the matrix  $A$  is positive semi-definite.

*Proof:* Let the coordinates of anchor  $i$ 's location be  $x_i$  and  $y_i$  with  $i = 1, 2, 3$ . Let

$$X = \begin{bmatrix} x_3 - x_2 & y_3 - y_2 \\ x_1 - x_3 & y_1 - y_3 \\ x_2 - x_1 & y_2 - y_1 \end{bmatrix} \quad (17)$$

Then it is easily verified that

$$A = 2XX^T \quad (18)$$

which is a positive semi-definite matrix. ■

If the distance information between sensor node 0 and  $r$  ( $r > 3$ ) anchor nodes are available to sensor 0, we can write down  $r - 2$  independent quadratic equality constraints. This can be obtained by demanding the coplanarity of the following node sets:  $\{0, 1, 2, 3\}, \{0, 1, 2, 4\}, \dots, \{0, 1, 2, r\}$ . Each coplanarity condition gives rise to a quadratic equality constraint in the following form through a procedure similar to that described above,

$$f_i(\varepsilon_1, \varepsilon_2, \varepsilon_i) = 0, \quad i = 3, 4, \dots, r \quad (19)$$

Further coplanarity constraints can be written down using other selections for 4 nodes, e.g.  $\{0, 2, 3, 4\}$ , but such constraints will not be independent of the set in (19).

### B. 3D Case

In 3D space, we consider the tetrahedron, as shown in Fig. 2, spanned by four anchor nodes 1, 2, 3 and 4, whose inaccurate distances relative to sensor node 0 are available to sensor 0. We assume the four anchor nodes are not in co-planar positions.

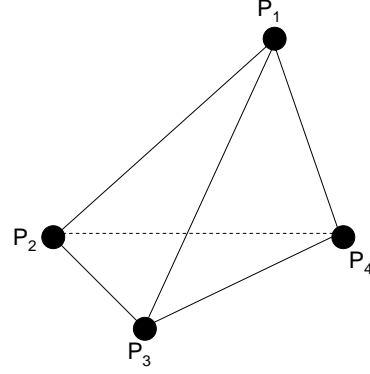


Fig. 2 Tetrahedron formed by four anchor nodes in 3D

Similarly to the 2D case, we have

$$D(\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = 0 \quad (20)$$

which defines a quadratic surface in  $\varepsilon_i$  with  $i = 1, 2, 3, 4$

$$\varepsilon^T \tilde{A} \varepsilon + \varepsilon^T \tilde{b} + \tilde{c} = 0 \quad (21)$$

After examining carefully each entry in matrix  $\tilde{A}$ , we define

$$\mathbf{Y} = \begin{bmatrix} (\mathbf{p}_2 - \mathbf{p}_4) \times (\mathbf{p}_3 - \mathbf{p}_4) & (\mathbf{p}_1 - \mathbf{p}_4) \times (\mathbf{p}_3 - \mathbf{p}_4) \\ (\mathbf{p}_1 - \mathbf{p}_4) \times (\mathbf{p}_2 - \mathbf{p}_4) & (\mathbf{p}_1 - \mathbf{p}_3) \times (\mathbf{p}_2 - \mathbf{p}_3) \end{bmatrix}^T \quad (22)$$

which is a 4-by-3 matrix, where “ $\times$ ” denotes the usual cross product of two vectors in 3D. Then

$$\tilde{A} = k\mathbf{Y} \cdot \mathbf{Y}^T \quad (23)$$

where  $k$  is a nonzero scaling factor. Hence, matrix  $\tilde{A}$  is also semi-definite.

Similar to the 2D case, if the distance information between sensor node 0 and  $s$  ( $s > 4$ ) anchor nodes are available to sensor 0, we can write down  $s - 3$  independent quadratic equality constraints.

## IV. AN OPTIMIZATION PROBLEM

Given all the algebraic constraints obtained in the last section, we now try to estimate the error in the inaccurate distances between sensor nodes and anchor nodes. One approach is to formulate the problem as a least squares problem to minimize the sum of the squared errors. Other objective functions are also adoptable depending on the specific application context. As discussed in [7], the least squares approach is sometimes not the most appropriate one to use. We use it here simply because of its clarity and simplicity of expression. The main point is that the quadratic constraints, once established, can be a powerful tool in various applications such as least square

optimizations. Here we use the 2D case to illustrate the least squares approach while 3D case can be dealt with by using the same procedures.

Let  $\varepsilon_i$  as defined in (5) be the error in the estimated squared distances between sensor 0 and anchor  $i$ . We want to minimize the sum of the squared errors

$$J = \varepsilon_1^2 + \varepsilon_2^2 + \dots + \varepsilon_r^2 \quad r \geq 3 \quad (24)$$

subject to  $r-2$  quadratic equality constraints as defined in equation (19).

When  $r=3$ , we have a least squares problem with one quadratic constraint, which is well studied [13], [14], [15]. When  $r > 3$ , we can use the following Lagrangian multiplier method.

Let  $\lambda_i$ ,  $i = 1, \dots, r-2$  be the Lagrangian multipliers. We can get the following Lagrangian multiplier form

$$H(\varepsilon_1, \dots, \varepsilon_r, \lambda_1, \dots, \lambda_{r-2}) = \sum_{i=1}^r \varepsilon_i^2 + \sum_{i=1}^{r-2} \lambda_i f_{i+2}(\varepsilon_1, \varepsilon_2, \varepsilon_{i+2}) \quad (25)$$

Because of the strict convexity of the function  $J$  and the positive semi-definiteness of the Hessians of functions  $f_{i+2}$ , when  $\lambda_i > 0$ , the Lagrangian  $H$  is a strictly convex function. Then there exists a unique global minimum. So numerical methods, such as gradient methods, can be exploited to search for the minimum. We will present as follows an analytical calculation by taking advantage of the special quadratic equality form of the constraints.

By differentiating the Lagrangian  $H$  with respect to  $\varepsilon_i$ ,  $i = 3, \dots, r$  and letting the result be zero, we have

$$\frac{\partial H}{\partial \varepsilon_i} = 2\varepsilon_i + \lambda_{i-2}(2d_{12}^2\varepsilon_i + 2(d_{12}^2 - d_{1i}^2 - d_{2i}^2)\varepsilon_1 + 2(d_{1i}^2 - d_{2i}^2 - d_{12}^2)\varepsilon_2 + 2d_{12}^2(d_{12}^2 - d_{1i}^2 - d_{2i}^2)) = 0 \quad (26)$$

By differentiating the Lagrangian  $H$  with respect to  $\lambda_i$ ,  $i = 1, \dots, r-2$  and letting the result be zero, we have

$$f_{i+2}(\varepsilon_1, \varepsilon_2, \varepsilon_{i+2}) = 0 \quad (27)$$

For each  $i = 1, \dots, r-2$ , equation (27) can be used to solve for  $\varepsilon_{i+2}$  in terms of  $\varepsilon_1$  and  $\varepsilon_2$ . And since equation (27) is quadratic in  $\varepsilon_{i+2}$ , it may lead to two possible solutions for  $\varepsilon_{i+2}$ . If we confine ourselves to the cases where only “robust” distances [7] are used in the localization calculation, then only one of the two solutions to this equation is desired. Consider the following geometrical explanation: the line connecting  $\mathbf{p}_1$  and  $\mathbf{p}_2$  divides the plane into two half planes; we discard the constraint  $f_{i+2} = 0$  if the corresponding anchor  $i+2$  and sensor 0 live in different half planes with respect to the line  $\mathbf{p}_1\mathbf{p}_2$ . This is required by the

“robustness” of the quadrilateral formed by sensor 0 and anchors 1, 2 and  $i+2$ . When all the anchors’ positions to be considered satisfy this robustness requirement, if (27) gives rise to two possible choices of  $\varepsilon_{i+2}$ , then the one that satisfies the geometric relation just described is what we want. From this, we have

$$\varepsilon_i = g_i(\varepsilon_1, \varepsilon_2) \quad \forall i \in \{3, \dots, r\} \quad (28)$$

By substituting (28) into (26), we have

$$\lambda_i = h_i(\varepsilon_1, \varepsilon_2) \quad \forall i \in \{1, \dots, r-2\} \quad (29)$$

By substituting (28) and (29) into  $\frac{\partial H}{\partial \varepsilon_1} = 0$  and  $\frac{\partial H}{\partial \varepsilon_2} = 0$ , we obtain two equations with two variables  $\varepsilon_1$  and  $\varepsilon_2$ . Hence, the Lagrangian multiplier method may find the minimum of the objective function (24).

In 3D, the same technique can be used to formulate an optimization problem with quadratic equality constraints. If a similar analytical approach is adopted, a subtle problem that arises is how to prune discrete feasible sets when each quadratic constraint function leads to two different solutions. A geometrical consideration might be helpful like that used in (27).

## V. CONCLUSIONS

This paper introduces the Cayley-Menger determinant as an important tool to formulating the geometric relations among nodes’ positions in sensor networks as quadratic constraints. It also discusses solutions to optimization problems to estimate the errors in the inaccurate distances between sensor nodes and anchor nodes. The solution of the optimization problem, when used to adjust noisy distance measurements, gives a set of distances between nodes which are completely consistent with the fact that sensor nodes live in the same plane or 3D-space as the anchor nodes. These technique serve as the foundation for most of the existing localization algorithms that depend on the sensors’ distances to anchors to compute each sensor’s location.

For future work, we will apply the technique presented in this paper to the existing localization algorithms and determine the most appropriate objective functions in the optimization process. Other optimization technique including semi-definite programming can also be exploited to accelerate the computation processes.

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