Proceedings of the 42nd IEEE
Conference on Decision and Control
Maui, Hawaii USA, December 2003

WeA01-1

The Multi-Agent Rendezvous Problem

J. Lin and A.S. Morse
Yale University
jie.lin, as.morse@yale.edu

B. D. O. Anderson
Australian National University and
National ICT Australia
brian.anderson@anu.edu.au

Abstract— This paper is concerned with the collective behavior of a group of $n > 1$ mobile autonomous agents, labelled 1 through n, which can all move in the plane. Each agent is able to continuously track the positions of all other agents currently within its "sensing region" where by an agent's _sensing region_ is meant a closed disk of positive radius r centered at the agent's current position. The _multi-agent rendezvous problem_ is to devise "local" control strategies, one for each agent, which without any active communication between agents, cause all members of the group to eventually rendezvous at single unspecified location. This paper describe two types of strategies for solving the problem. The first consists of agent strategies which are mutually synchronized in the sense that all depend on a common clock. The second consists of strategies which can be implemented independent of each other, without reference to a common clock.

Current interest in cooperative control has led to the development of a number of distributed control algorithms capable of causing large groups of mobile autonomous agents to perform useful tasks. Of particular interest here are provably correct algorithms which solve what we shall refer to as the "multi-agent rendezvous problem." This problem, which was posed in [1], is concerned with the collective behavior of a group of $n > 1$ mobile autonomous agents, labelled 1 through n, which can all move in the plane. Each agent is able to continuously track the positions of all other agents currently within its "sensing region" where by an agent's *sensing region* is meant a closed disk of positive radius r centered at the agent's current position. The *multi-agent rendezvous problem* is to devise "local" control strategies, one for each agent, which without any active communication between agents, cause all members of the group to eventually rendezvous at single unspecified location.

In this paper, as in [1], we consider distributed strategies which guide each agent toward rendezvous by performing a sequence of "stop-and-go" maneuvers. A *stop-and-go maneuver* takes place within a time interval consisting of two consecutive sub-intervals. The first, called a *sensing period*, is an interval of fixed length during which the agent is stationary. The second, called a *maneuvering period*, is an interval of variable length during which the agent moves from its current position to its next 'way-point' and again come to rest. Successive way-points for each agent are chosen to be within r_M units of each other where r_M is a pre-specified positive distance no larger than r. It is assumed that there has been chosen for each agent i, a positive number τ_{M_i}, called a *maneuver time*, which is large enough so that the required maneuver for agent i from any one way-point to the next can be accomplished in at most τ_{M_i} seconds. Since our interest here is exclusively with devising of *high level* strategies which dictate when and where agents are to move, we shall not deal with how maneuvers are actually carried out or with how vehicle collisions are to be avoided.

This research was supported by the National Science Foundation and by the Australia Government Department of Communications, Information Technology and the Arts and the Australian Research Council through the Centre of Excellence Program

In the sequel we describe two families of stop-and-go strategies. The first, which includes the specific strategies proposed in [1], consists of agent strategies which are mutually synchronized in the sense that all depend on a common clock. The second consists of strategies which can be implemented independent of each other, without reference to a common clock. We begin with the synchronous case.

I. Synchronous Case

In the synchronous case, the maneuvering times for all agents are all the same length positive value τ_M. Along any trajectory of the system to be considered, the real time axis can be partitioned into a sequence of time intervals $[0, t_1), [t_1, t_2), \dots [t_{k-1}, t_k), \dots$, each of length at least τ_M. Each interval consists of a sensing period followed by a maneuvering period of fixed length τ_M. All agents function in synchronization in the sense that all are at rest during sensing periods and all can maneuver only during maneuvering periods. In particular, all agents actions are synchronized to the time sequence $\bar{t}_1, \bar{t}_2, \dots \bar{t}_k \dots$ where $\bar{t}_k$ denotes the real time $t_k - \tau_M$ at which the kth maneuvering period begins. Agent i's *registered neighbors* at each time during its kth maneuvering period $[\bar{t}_k, t_k)$, are those agents, except for agent i, which are within agent i's sensing region at time $\bar{t}_k$. Note that this definition is a symmetric relation on the set of all agents; i.e., if agent i is a registered neighbor of agent j during maneuvering period k, then agent j is a neighbor of agent i during the same maneuvering period. As we shall see, special steps will have to be taken to achieve a similar property in the asynchronous case.

A pair of agents which are registered neighbors during maneuvering period k are said to satisfy the *pairwise motion constraint* during the period if the positions to which they move at time t_k are both within an closed disk of diameter r centered at the mean of their registered positions at time $\bar{t}_k$. The definition implies that any two agents which are registered neighbors during maneuvering period k will be registered neighbors during maneuvering period $k+1$ if they satisfy the pairwise motion constraint during the kth. We are interested in strategies possessing this property and accordingly make the following assumption.

Cooperation Assumption: During each maneuvering period k, each pair of registered neighbors restrict their motions to satisfy the pairwise motion constraint.

Agent i's *kth way-point* is the point to which agent i is to move to at time t_k. Thus if $x_i(t)$ denotes the position of agent i at time t represented in a world coordinate system, then $x_i(t_k)$ and agent i's kth way-point are one and the same. The rule which determines each such way-point is a function depending only on the number and relative positions of agent i's registered neighbors. In particular, if agent i has m_i registered neighbors at time $\bar{t}_k$, positioned relative to agent i at points

$$z_j \triangleq x_{i_j}(\bar{t}_k) - x_i(\bar{t}_k),\ j \in \{1, 2, \dots, m_i\} \tag{1}$$

then agent i's kth way-point is

$$x_i(t_{k-1}) + u_{m_i}(z_1, z_2, \dots, z_{m_i}) \tag{2}$$

where $u_0 = 0$, $u_m : \mathbb{D}^m \to \mathbb{D}_M$, $m \in \{1, \dots, n-1\}$, and $\mathbb{D}$ and $\mathbb{D}_M$ are the closed disks of radii r and r_M respectively, centered at the origin in $\mathbb{R}^2$. In other words, if agent i has no registered neighbors at time $\bar{t}_k$, {i.e., $m_i = 0$}, it does not move during the kth maneuvering period. On the other hand, if agent i has $m_i > 0$ neighbors at time $\bar{t}_k$ with relative positions $z_1, z_2, \dots, z_{m_i}$, then agent i moves to the position $x_i(t_{k-1}) + u_{m_i}(z_1, z_2, \dots, z_{m_i})$ at time t_k. Thus

$$\begin{aligned} x_i(t_k) &= x_i(t_{k-1}) + u_{m_i(\bar{t}_k)}(x_{i_1}(\bar{t}_k) - x_i(\bar{t}_k),\ x_{i_2}(\bar{t}_k) \\ &\quad -x_i(\bar{t}_k),\ \dots,\ x_{i_{m_i(\bar{t}_k)}}(\bar{t}_k) - x_i(\bar{t}_k)) \end{aligned} \tag{3}$$

In the sequel we will explain how the u_m are defined. At the very least we will require each to be a continuous function.

A. *Definition of u_m*

We've already defined $u_0 = 0$. To define u_m for $m > 0$ it is necessary to take into account the pairwise motion constraint. Toward this end, for each $z \in \mathbb{D}$, let $\mathcal{C}(z)$ denote the closed disk of diameter r centered at the point $\frac{1}{2}z$. More generally, for each $\{z_1, z_2, \dots, z_m\} \in \mathbb{D}^m$, let

$$\mathcal{C}(z_1, z_2, \dots, z_m) = \bigcap_{j=1}^{m} \mathcal{C}(z_j) \tag{4}$$

Note that 0 is in each $\mathcal{C}(z_i)$ and moreover that each such $\mathcal{C}(z_i)$ is closed and strictly convex. Consequently $\mathcal{C}(z_1, z_2, \dots, z_m)$ is either the singleton $\{0\}$ or a strictly convex, closed set containing 0. We can now define u_m to be any continuous function on $\mathbb{D}^m$ satisfying

$$\begin{aligned} u_m(z_1, z_2, \dots, z_m) \in\ & \mathbb{D}_M \cap \mathcal{C}(z_1, z_2, \dots, z_m) \\ & \cap \langle 0, z_1, z_2, \dots, z_m \rangle \end{aligned} \tag{5}$$

for all $\{z_1, z_2, \dots, z_m\} \in \mathbb{D}^m$ where $\langle 0, z_1, z_2, \dots, z_m \rangle$ is the convex hull of the points $0, z_1, z_2, \dots, z_m$. The u_m are further required to have the property that

$$u_m(z_1, z_2, \dots, z_m) \neq \text{a corner of } \langle 0, z_1, z_2, \dots, z_m \rangle \tag{6}$$

unless $z_1 = z_2 = \cdots = z_m = 0$. In other words, u_m is required to be (i) a continuous function on $\mathbb{D}^m$ which maps each $\{z_1, z_2, \dots, z_m\} \in \mathbb{D}^m$ into $\mathbb{D}_M \cap \mathcal{C}(z_1, z_2, \dots, z_m) \cap \langle 0, z_1, z_2, \dots, z_m \rangle$ and (ii) a function with the property that $u_m(z_1, z_2, \dots, z_m)$ is not a corner of $\langle 0, z_1, z_2, \dots, z_m \rangle$ unless $z_1 = z_2 = \cdots = z_m = 0$. Examples of functions satisfying these conditions will be given in the sequel.

One way to go about defining specific u_m which are continuous and which satisfy these requirements, is by first defining what we shall refer to as a target point. By a *target point* is meant a continuous function $\tau : \mathbb{D}^m \to \langle 0, z_1, z_2, \dots, z_m \rangle$ defined in such a way so that for each $\{z_1, z_2, \dots, z_m\} \in \mathbb{D}^m$ for which 0 is a corner of $\langle 0, z_1, z_2, \dots, z_m \rangle$, the segment of the line from 0 to $\tau(z_1, z_2, \dots, z_m)$ which lies within $\mathcal{C}(z_1, z_2, \dots, z_m)$ has positive length. For should it be possible to define such a τ, one could satisfy (5) and (6) as well as the continuity requirement with a control of the form

$$u_m = g(z_1, z_2, \dots, z_m)\tau(z_1, z_2, \dots, z_m)$$

where $g : \mathbb{D}^m \to \mathbb{R}$ is any continuous, positive definite function satisfying

$$g < \max_{(0,1]}\{\mu : \mu\tau \in \mathbb{D}_M \bigcap \mathcal{C}(z_1, z_2, \dots, z_m)\}$$

Note that $g\tau \in \langle 0, z_1, z_2, \dots, z_m \rangle$, $\forall g \in [0, 1]$ because $0 \in \langle 0, z_1, z_2, \dots, z_m \rangle$. The role of g is therefore to scale down the magnitude of τ enough to insure that $g\tau$ is in the constraint set $\mathbb{D}_M \bigcap \mathcal{C}(z_1, z_2, \ldots, z_m)$.

It might be thought that one could choose for τ, the centroid of $\langle 0, z_1, z_2, \ldots, z_m \rangle$ or perhaps the average of the z_i and 0, namely

$$\tau \triangleq \frac{1}{m+1} \sum_{i=1}^{m} z_i,$$

Both candidate definitions satisfy the requirement that $\tau(z_1, z_2, \ldots, z_m)$ must be a point in $\langle 0, z_1, z_2, \ldots, z_m \rangle$. Unfortunately, simple examples show that centroid definition does not necessarily yield a function which satisfies the continuity requirement while the averaging definition may lead to a function which fails to satisfy the requirement that when 0 is a corner of $\langle 0, z_1, z_2, \ldots, z_m \rangle$, the segment of the line from 0 to $\tau(z_1, z_2, \ldots, z_m)$ which lies within $\mathcal{C}(z_1, z_2, \ldots, z_m)$ has positive length. For example, the centroid of the convex hull of the points $(0,0)$, $z_1 = (0,1)$ and $z_2 = (p,1)$ is at $(\frac{p}{3}, \frac{2}{3})$ for $p > 0$ and at $(0, \frac{1}{2})$ for $p = 0$ so the centroid is discontinuous at $p = 0$. As a counterexample to the use of coordinate averaging to define a target point, note that the average of the four points located at $(0,0)$, $z_1 = (-r, 0)$, $z_2 = (\frac{2r}{3}, \frac{r}{2})$, and $z_3 = (\frac{r}{3}, \frac{r}{2})$ is at $(0, r)$ while the constraint set $\mathcal{C}(z_1, z_2, z_3)$ determined by these points must be contained in the constraint disk $\mathcal{C}(z_1)$. Since the line $\mathcal{L}$ from $(0,0)$ to $(0,r)$ is tangent to this disk at the origin, the intersection of $\mathcal{L}$ with $\mathcal{C}(z_1, z_2, z_3)$ is just the point $(0,0)$ and consequently not a line segment of positive length.

In the sequel we shall approach the problem of defining of τ in a slightly different way. We begin by stating the following proposition which provides a simple condition on $\tau(\cdot)$, which if satisfied, automatically implies satisfaction of the requirement that when 0 is a corner of $\langle 0, z_1, z_2, \ldots, z_m \rangle$, the segment of the line from 0 to $\tau(z_1, z_2, \ldots, z_m)$ which lies within $\mathcal{C}(z_1, z_2, \ldots, z_m)$ has positive length.

Proposition 1: Let $z_1, z_2, \ldots, z_m$ be a set of $m > 0$ points in $\mathbb{D}$ which are not all 0. If 0 is a corner of $\langle 0, z_1, z_2, \ldots, z_m \rangle$ and z is any non-zero point in $\mathbb{D}$ within r units of each point in $\{z_1, z_2, \ldots, z_m\}$, then the segment of the line from 0 to z which lies in $\mathcal{C}(z_1, z_2, \ldots, z_m)$, has positive length.

Proposition 1 suggest the following approach for defining a target point. First, for each $z \in \mathbb{D}$, let $\mathcal{D}(z)$ denote a closed disk of radius r centered at z. More generally for any set of $m > 0$ point $z_1, z_2, \ldots, z_m$ in $\mathbb{D}$, write

$$\mathcal{D}(z_1, z_2, \ldots, z_m) = \bigcap_{i=1}^{m} \mathcal{D}(z_i)$$

By construction, each point in $\mathcal{D}(z_1, z_2, \ldots, z_m)$ is within r units of each point in $\{z_1, z_2, \ldots, z_m\}$. Thus $0 \in \mathcal{D}(z_1, z_2, \ldots, z_m)$ because $z_i \in \mathbb{D}$, $i \in \{1, 2, \ldots, m\}$.

Second, note that if $z_1, z_2, \ldots, z_m$ is any set of $m > 0$ points in $\mathbb{D}$ which are not all zero and for which 0 is a corner of $\langle 0, z_1, z_2, \ldots, z_m \rangle$, then by Proposition 1 the segment of the line from 0 to any non-zero point $\mathbb{D} \cap \mathcal{D}(z_1, z_2, \ldots, z_m)$ which lies in $\mathcal{C}(z_1, z_2, \ldots, z_m)$, must have positive length. It follows that any continuous function $\tau : \mathbb{D}^m \rightarrow \langle 0, z_1, z_2, \ldots, z_m \rangle$ which satisfies $\tau(z_1, z_2, \ldots z_m) \in \mathbb{D} \bigcap \mathcal{D}(z_1, z_2, \ldots, z_m) \bigcap \langle 0, z_1, z_2, \ldots, z_m \rangle$ and which is non-zero whenever 0 is a corner of $\langle 0, z_1, z_2, \ldots, z_m \rangle$ and $z_1, z_2, \ldots, z_m$ are not all zero, fulfills all the conditions required to be a target point. In the sequel we will show that there are at least two different ways to so define τ.

1) The centroid of $\mathbb{D} \cap \mathcal{D}(z_1, z_2, \ldots, z_m)$*:* In order for the centroid of $\mathbb{D} \cap \mathcal{D}(z_1, z_2, \ldots, z_m)$ to be a target point, it must depend continuously on the z_i and, in addition, must have the property that it is non-zero for any set of m points in $\mathbb{D}$ which are not all zero and for which 0 is a corner of $\langle 0, z_1, z_2, \ldots, z_m \rangle$. These properties are shown to hold in the full-length version of this paper. The continuity of the centroid of $\mathbb{D} \cap \mathcal{D}(z_1, z_2, \ldots, z_m)$ proves to depend crucially on the fact that the centroid is at 0 whenever the area of $\mathbb{D} \cap \mathcal{D}(z_1, z_2, \ldots, z_m)$ is zero. This property is not shared by the centroid of $\langle 0, z_1, z_2, \ldots, z_m \rangle$ and it is for this reason that the centroid of $\langle 0, z_1, z_2, \ldots, z_m \rangle$ is not a continuous function of the z_i.

2) The center of the smallest circle containing $\langle 0, z_1, z_2, \ldots, z_m \rangle$*:* It is also possible to define τ to be the center of the smallest circle containing $\langle 0, z_1, z_2, \ldots, z_m \rangle$. To understand why this is so, let us note first that for any set of points $z_i \in \mathbb{D}$, $i \in \{1, 2, \ldots, m\}$, the set of points $\mathcal{Q} \triangleq \{0, z_1, \ldots, z_m\}$ is contained in a circle of radius r. It follows that the center of this circle, which is at 0, is at most r units from every other point. This suggests that one might choose for $\tau(z_1, z_2, \ldots, z_m)$ the center $\tau_C(z_1, z_2, \ldots z_m)$ of the smallest circle containing $\mathcal{Q}$ or equivalently $\langle 0, z_1, z_2, \ldots, z_m \rangle$, since $\tau_C(z_1, z_2, \ldots, z_m)$ would have to be within r units of every point in $\mathcal{Q}$. It is known that there is such a smallest circle [2] and that if the z_i are not all zero, $\tau_C(z_1, z_2, \ldots z_m)$ is either the midpoint between two of the points in $\mathcal{Q}$ or a point within the interior of a triangle formed from at least one set of three points in $\mathcal{Q}$ [1]. In either case it is clear that $\tau_C(z_1, z_2, \ldots z_m) \in \langle 0, z_1, z_2, \ldots, z_m \rangle$ and, if the z_i are not all zero and 0 is a corner of $\langle 0, z_1, z_2, \ldots, z_m \rangle$, that $\tau_C(z_1, z_2, \ldots z_m)$ is nonzero as well. Furthermore it can be shown that $\tau_C(z_1, z_2, \ldots z_m)$ depends continuously on the z_i [3]. In other words, $\tau_C(z_1, z_2, \ldots z_m)$ satisfies all the conditions required to be a target point. This elegant choice for τ is the one proposed in [1].

B. Main Results

Note that because agents don't move during sensing periods, the position of each agent at time t_{k-1} is the same as its position at time $\bar{t}_k$. Thus (3) can be re-written as

$$\begin{aligned} x_i(t_k) &= x_i(t_{k-1}) + u_{m_i(t_{k-1})}(x_{i_1}(t_{k-1}) - x_i(t_{k-1}), \\ &\quad x_{i_2}(t_{k-1}) - x_i(t_{k-1}), \ \ldots, \ x_{i_{m_i(t_{k-1})}}(t_{k-1}) \\ &\quad -x_i(t_{k-1})) \end{aligned} \tag{7}$$

where $m_i(t_{k-1}) \triangleq m_i(\bar{t}_k)$ Because of this, the system just defined admits the model of a nonlinear discrete-time system with state $x(t_k)$ = column $\{x_1(t_k), x_2(t_k), \ldots x_n(t_k)\}$ evolving on the time set $t_0, t_1, \ldots t_k, \ldots$. Analysis of this system depends on the relationships between neighbors and how they evolve with time. These relationships can be conveniently described by a simple, undirected graph with vertex set $\{1, 2, \ldots, n\}$ which is defined so that (i, j) is one of the graph's edges just in case agents i and j are registered neighbors during maneuvering period k. Since these relationships can change from one maneuvering period to the next, so can the graph which describes them. In the sequel we use the symbol $\mathcal{P}$ to denote a suitably defined set, indexing the class of all simple graphs $\mathbb{G}_p$ on n vertices. Let us partially order the set $\{G_p : p \in \mathcal{P}\}$ by agreeing to say that $\mathbb{G}_p$ is contained in $\mathbb{G}_q$ if the edge set of $\mathbb{G}_p$ is a subset on the edge set of $\mathbb{G}_q$. It is natural then to define the *union* of a collection of such graphs, $\{\mathbb{G}_{p_1}, \mathbb{G}_{p_2}, \ldots, \mathbb{G}_{p_m}\}$, to be the simple graph $\mathbb{G}$ with vertex set $\{1, 2, \ldots, n\}$ and edge set equaling the union of the edge sets of all of the graphs in the collection.

Let $\sigma(k)$ denote the index of the graph in $\{G_p : p \in \mathcal{P}\}$ which describes the relationship between registered neighbors during maneuvering period k. Because of the cooperation assumption, we know that each agent keeps all of its registered neighbors as the system evolves. What this means is the sequence of graphs $\mathbb{G}_{\sigma(1)}, \mathbb{G}_{\sigma(2)}, \ldots, \mathbb{G}_{\sigma(k)}, \ldots$ forms the ascending chain

$$\mathbb{G}_{\sigma(1)} \subset \mathbb{G}_{\sigma(2)} \subset \cdots \mathbb{G}_{\sigma(k)} \cdots \tag{8}$$

Because $\{G_p : p \in \mathcal{P}\}$ is a finite set, the chain must converge to the graph

$$\mathbb{G} \triangleq \bigcup_{k=1}^{\infty} \mathbb{G}_{\sigma(k)} \tag{9}$$

in a finite number of steps. Since the sequence of graphs stops changing in a finite number of steps, rendezvousing at a single point can only occur if $\mathbb{G}$ is a complete graph. There is however, no a priori guarantee that along a particular trajectory, $\mathbb{G}$ will turn out to be complete. On the other hand, it is clear that $\mathbb{G}$ will always be at least connected if the initial graph $\mathbb{G}_{\sigma(1)}$ in the ascending chain is. It turns out that connectivity of $\mathbb{G}_{\sigma(1)}$ implies not only that $\mathbb{G}$ is connected but also that the types of distributed control strategies just described actually cause all agents to rendezvous at a single point.

Theorem 1: Let $u_0 = 0 \in \mathbb{D}_M$ and for each $m \in \{1, 2, \ldots, n-1\}$, let $u_m : \mathbb{D}^m \rightarrow \mathbb{D}_M$ be any continuous function satisfying (5) and (6). For each set of initial agent positions $x_1(0), x_2(0), \ldots, x_n(0)$, each agent's position $x_i(t)$ converges to a unique point $y_i \in \mathbb{R}^2$ such that for each $i, j \in \{1, 2, \ldots, n\}$, either $y_i = y_j$ or $||y_i - y_j|| > r$. Moreover, if agents i and j are registered neighbors at any time t, then $y_i = y_j$.

Theorem 1 states that the strategies under consideration cause all agents positions to converge to points in the plane with the property that each two such points are either equal to each other, or separated by a distance greater than r units. The theorem further states that if two agents are ever registered neighbors of each other, then their positions converge to the same point. We are led to the following corollary.

Corollary 1: If the graph characterizing registered neighbors during maneuvering period 1 is connected, then the positions of all n agents converge to a common point in the plane.

It is quite straight forward to extend these results to the leader-follower case when the rendezvous point is specified at the outset. This can be accomplished by simply fixing one additional agent {i.e., a virtual agent} at the desired rendezvous point and letting the remaining n agents maneuver just as before. With initial graph connectivity of all $n+1$ agent positions, convergence to the position of the virtual agent is then assured.

A more interesting case occurs when two virtual agents are fixed at distinct points in the plane. In this case it can be shown that with initial connectivity of the $n+2$ - agent graph, all n agents will eventually move to positions on the line connecting the two virtual agents and will distribute themselves in a predictable manner depending only the number of agents, r and the distance between the two fixed, virtual agents. This behavior will be explored in greater depth in another paper dealing with forming formations using distributed control.

1) Trapping: While the graph connectivity hypothesis of Corollary 1 is sufficient for rendezvousing, it is not necessary. For example, suppose that the $\mathbb{G}_{\sigma(1)}$ has a connected component $\mathbb{G}_C$ which contains a simple closed cycle whose vertices are $i_1, i_2, \ldots, i_m$. Then in the plane, the geometric form obtained by connecting by a straight line, the initial position of each agent $i_j \in \{i_1, i_2, \ldots, i_m\}$ with its registered neighbors with labels in $\{i_1, i_2, \ldots, i_m\}$, will be a simple, closed, polygon $\mathbb{P}$. It turns out that if the initial positions of all agents whose labels are not in the vertex set of $\mathbb{G}_C$, are within $\mathbb{P}$, then rendezvous will necessarily occur. While this conclusion might appear to be an obvious consequence of the established property that agents $i_j \in \{i_1, i_2, \ldots, i_m\}$ eventually rendezvous at a point, actually proving that this is so is not so straight forward. There are two reasons for this. First there is no guarantee that the polygon $\mathbb{P}(t)$ formed by the positions at time t of agents $i_j \in \{i_1, i_2, \ldots, i_m\}$ will remain simple as the system evolves, even if it is initially; thus just what it means for an agent to be "inside" of $\mathbb{P}(t)$ requires a more sophisticated notion of interior than the obvious one for a simple closed curve in the plane and this in turn complicates the analysis. Second, it is quite possible that an agent initially positioned inside of $\mathbb{P}(0)$, will be outside of $\mathbb{P}(t)$ for some $t > 0$. In the sequel we explain how to overcome both of these difficulties and in so doing we establish a rendezvousing result along the lines just described.

We begin by reviewing the concept of a "winding number" and what it means for a point to be inside of a closed curve in $\mathbb{R}^2$. Let $\kappa : [0,1] \rightarrow \mathbb{R}^2$ be any continuous closed curve and let y be any point in $\mathbb{R}^2$ which does not lie on κ. The *winding number* of y with respect to κ, written $\text{wn}(\kappa, y)$, is the number of times a point p traversing κ encircles y in a counter-clockwise direction as p makes a full circuit of κ. Points not on κ with non-zero winding numbers are inside of κ while those with a winding number of zero are outside of κ. There is a well-known formula for $\text{wn}(\kappa, y)$, involving the integral around a closed contour $\tilde{\kappa} : [0,1] \rightarrow \mathbb{C}$ in the complex plane [4]. $\tilde{\kappa}$ is a representation of κ resulting from the assignment to each vector $x = [a \quad b]'$ in $\mathbb{R}^2$, the associated complex number $\tilde{x} \triangleq a + jb$. In this setting, $\text{wn}(\kappa, y)$ is given by the contour integral

$$\text{wn}(\kappa, y) = \frac{1}{2\pi j} \oint_{\tilde{\kappa}} \frac{dz}{z - \tilde{y}}$$

The closed curves of interest here are of a specific type determined by finite point sets in $\mathbb{R}^2$. In particular, let us note that any ordered set of $m > 0$ points $\{y_1, y_2, \ldots, y_m\}$ in $\mathbb{R}^2$ uniquely determines a continuous, piecewise-linear, closed curve $c : [0, m] \rightarrow \mathbb{R}^2$ defined so that $c(t) = (t + 1 - i)y_{i+1} + (i - t)y_i, \quad i - 1 \leq t \leq i,\ i \in \{1, 2, \ldots, m\}$ where $y_{m+1} = y_1$. An ordered set $\{y_1, y_2, \ldots, y_m\}$ of three or more such points is called a *cycle* if $\|y_{i+1} - y_i\| \leq r,\ i \in \{1, 2, \ldots, m\}$; in the sequel we denote such a cycle by $\lceil y_1, y_2, \ldots, y_m \rceil$. A point $z \in \mathbb{R}^2$ is called an *interior point* of $\lceil y_1, y_2, \ldots, y_m \rceil$ if it is an interior point of the closed, piece-wise linear curve c determined by $\{y_1, y_2, \ldots, y_m\}$. A point $z \in \mathbb{R}^2$ is said to be *connected* to non-empty set of vectors $\{y_1, y_2, \ldots, y_m\}$ through a set of vectors $\{x_1, x_2, \ldots, x_n\}$ in $\mathbb{R}^2$ if there exists a subset $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\}$ with $x_{i_k} \in \{y_1, y_2, \ldots, y_m\}$ such that $\|z - x_{i_1}\| \leq r$ and $\|x_{i_{s-1}} - x_{i_s}\| \leq r,\ i \in \{2, 3, \ldots k\}$. The following corollary to Theorem 1 is the main result on trapping.

Corollary 2: Suppose that the set of initial positions $\{x_1(0),\ x_2(0),\ \ldots,\ x_n(0)\}$ of the n agents contains a cycle $\lceil x_{i_1}(0), x_{i_2}(0), \ldots, x_{i_m}(0) \rceil$. Then all agents with positions initially connected to the cycle through $\{x_1(0),\ x_2(0),\ \ldots,\ x_n(0)\}$ eventually rendezvous at one point with all agents initially positioned inside the cycle.

II. Asynchronous Case

The strategy described in the previous section cannot be regarded as truly distributed because each agent's decisions must be synchronized to a common clock shared by all other agents in the group. In this section we redefine the strategies so that a common clock is not required. To do this it will be necessary to modify somewhat what is meant by a registered neighbor and by a registered neighbor's position.

In the asynchronous case, for each agent i, the real time axis can be partitioned into a sequence of time intervals $[0, t_{i1}),\ [t_{i1}, t_{i2}),\ \ldots,\ [t_{i(k_i-1)}, t_{ik_i}),\ \ldots$ each of length at most $\tau_D + \tau_{M_i}$ where τ_D is a number greater than τ_{M_i} called a *dwell time*. Each interval $[t_{i(k_i-1)}, t_{ik_i})$ consists of a *sensing period* $[t_{i(k_i-1)}, \bar{t}_{ik_i})$ of fixed length τ_D during which agent i is stationary, followed by a maneuvering period $[\bar{t}_{ik_i}, t_{ik_i})$ of length at most τ_{M_i} during which agent i moves from its current position to its next way-point. It is assumed that during each of its maneuvering periods, an agent keeps moving except possibly for brief periods which are each shorter than a pre-specified positive time τ_S called a *sensing time*. For reasons to be made clear below, we shall require τ_S to satisfy

$$\tau_S \leq \frac{1}{2}(\tau_D - \tau_{M_i}) \qquad \forall i \in \{1, 2, \ldots, n\} \tag{10}$$

Although all agents use the same dwell time, they function asynchronously in the sense that the time sequences $t_{i1}, t_{i2}, \cdots,\ i \in \{1, 2, \ldots, n\}$ are uncorrelated. Thus each agent's strategy can be implemented independent of the rest, without the need for a common clock.

For the asynchronous case, agent i's *registered neighbors* at each time during its k_ith maneuvering period $[\bar{t}_{ik_i}, t_{ik_i})$ are taken to be those agents which are fixed at one position within agent i's sensing region for at least τ_S seconds during agent i's k_ith sensing period $[t_{i(k_i-1)}, \bar{t}_{ik_i})$. If agent j is such a registered neighbor of agent i, there may be several distinct intervals of length at least τ_S within sensing period k_i during which agent j is stationary. Let t^* denote the end time of the last of these. For purposes of calculation, agent i takes the *registered position* of agent j during maneuver period k_i, to be the actual position of agent j at time t^*.

Note that constraint (10) implies that each of agent i's sensing periods must overlap either one or two of agent j's sensing periods and moreover that at least one of these overlaps must be at least τ_S units long. Thus for fixed i and any j, t^* is well-defined on each of agent i's sensing intervals. It follows that agent i's registered neighbors and registered neighbor positions are also well-defined on each of agent i's maneuvering periods. Note in addition that if agent j is a registered neighbor of agent i on agent i's k_ith maneuvering period, then agent i must be a registered neighbor of agent j on agent j's $k_j(t^*)$ maneuvering period where for all j, $k_j : [0, \infty) \rightarrow \{0, 1, 2, \dots, \}$ is that function which assigns to t, the index k_j of the interval $[t_{j(k_j-1)}, t_{jk_j})$ which contains t. Since t^* is a well-defined function of i, k_i and j, in the sequel we sometimes write $t^*(i, k_i, j)$ for t^*.

Agent i is said to satisfy the *motion constraint induced by registered neighbor* j during maneuver period k_i if agent j is a registered neighbor of agent i during the period and if the position to which agent i moves at time t_{ik_i} is within an closed disk of diameter r centered at the mean of agent i's position at the beginning of the period and the registered position of agent j at the beginning of the period. The following lemma is key.

Lemma 1: Suppose that agent i satisfies the motion constraint induced by registered neighbor j during maneuver period k_i and that agent j correspondingly satisfies the motion constraint induced by registered neighbor i during maneuver period $k_j(t^*(i, k_i, j))$. Then agent j will be a registered neighbor of agent i during maneuvering period $k_i + 1$ and agent i will be a registered neighbor of agent j during maneuvering period $k_j(t^*(i, k_i, j)) + 1$.

We make the following assumption.

Cooperation Assumption: Each agent i satisfies the motion constraints induced by each of its registered neighbors during each of its maneuvering periods.

Suppose that the cooperation assumption is satisfied. Lemma 1 implies that if agent j is a registered neighbor of agent i during maneuvering interval k_i then it will also be a registered neighbor of agent i during maneuvering interval $k_i + 1$. In other words, if the cooperation assumption is satisfied, each agent retains all of its prior registered neighbors as the system evolves.

Just like the synchronous case, agent i's k_ith *way-point* is the point to which agent i is to move to at the end of its k_ith maneuvering period. Thus if $x_i(t)$ denotes the position of agent i at time t represented in a world coordinate system, then $x_i(t_{ik_i})$ and agent i's k_ith way-point are one and the same. The rule which determines each such way-point is essentially the same as in the synchronous case, except that now it is a function depending only on its own position at the beginning of its k_ith maneuvering period and the positions of its registered neighbors at the beginning of the period. In particular, if agent i has m_i registered neighbors at the beginning of its k_ith maneuvering period, with registered positions relative to agent i at points $z_1, z_2, \dots, z_m$, then agent i's k_ith way-point is

$$x_i(t_{k_i}) = x_i(t_{k_i-1}) + u_m(z_1, z_2, \dots, z_m) \qquad (11)$$

where u_m is defined exactly as before.

III. Concluding Remarks

It turns out that the asynchronous system just described admits the model of a hybrid dynamical system in which *non-deterministic* state transitions can occur. This somewhat surprising property stems from the fact that low-level individual vehicle maneuvering is {intentially} not modelled. Nonetheless, in the full-length version of this paper it is shown that rendezvousing is achieved in the asynchronous case provided a suitably defined "registered neighbor graph" similar to the graph defined in the synchronous case, is initially connected.

IV. References

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