Is Monopoli's Model Reference Adaptive Controller Correct? *

A. S. Morse
Center for Computational Vision and Control
Department of Electrical Engineering
Yale University, New Haven, CT 06520 USA

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In 1974 R. V. Monopoli published a paper [1] in which he posed the now classical model reference adaptive control problem, proposed a solution and presented arguments intended to establish the solution's correctness. Subsequent research [2] revealed a flaw in his proof which placed in doubt the correctness of the solution he proposed. Although provably correct solutions to the model reference adaptive control problem now exist {see [3] and the references therein}, the problem of deciding whether or not Monopoli's original proposed solution is in fact correct remains unsolved. The aim of this note is to review the formulation of the classical model reference adaptive control problem, to describe Monopoli's proposed solution, and to outline what's known at present about its correctness.

1 The Classical Model Reference Adaptive Control Problem

The *classical model reference adaptive control problem* is to develop a dynamical controller capable of causing the output y of an imprecisely modelled siso process $\mathbb{P}$ to approach and track the output y_{ref} of a pre-specified reference model $\mathbb{M}_{\text{ref}}$ with input r. The underlying assumption is that the process model is known only to the extent that it is one of the members of a pre-specified class $\mathcal{M}$. In the classical problem $\mathcal{M}$ is taken to be the set of all siso controllable, observable linear systems with strictly proper transfer functions of the form $g\frac{\beta(s)}{\alpha(s)}$ where g is a nonzero constant called the *high frequency gain* and $\alpha(s)$ and $\beta(s)$ are monic, coprime polynomials. All g have the same sign and each transfer function is *minimum phase* {i.e., each $\beta(s)$ is stable}. All transfer functions are required to have the same relative degree $\bar{n}$ {i.e., $\deg \alpha(s) - \deg \beta(s) = \bar{n}$.} and each must have a McMillan degree not exceeding some pre-specified integer n {i.e., $\deg \alpha(s) \leq n$}. In the sequel we are going to discuss a simplified version of the problem in which all $g = 1$ and the reference model transfer function is of the form $\frac{1}{(s+\lambda)^{\bar{n}}}$ where λ is a positive number. Thus $\mathbb{M}_{\text{ref}}$ is a system of the form

$$\dot{y}_{\text{ref}} = -\lambda y_{\text{ref}} + \bar{c}x_{\text{ref}} + \bar{d}r \qquad \dot{x}_{\text{ref}} = \bar{A}x_{\text{ref}} + \bar{b}r \tag{1}$$

where $\{\bar{A},\ \bar{b},\ \bar{c},\ \bar{d}\}$ is a controllable, observable realization of $\frac{1}{(s+\lambda)^{(\bar{n}-1)}}$.

2 Monopoli's Proposed Solution

Monopoli's proposed solution is based on a special representation of $\mathbb{P}$ which involves picking any n-dimensional, single-input, controllable pair (A, b) with A stable. It is possible to prove [1, 4] that

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the assumption that the process $\mathbb{P}$ admits a model in $\mathcal{M}$, implies the existence of a vector $p^* \in \mathbb{R}^{2n}$ and initial conditions $z(0)$ and $\bar{x}(0)$, such that u and y exactly satisfy

$$\begin{aligned}
\dot{z} &= \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} z + \begin{bmatrix} b \\ 0 \end{bmatrix} y + \begin{bmatrix} 0 \\ b \end{bmatrix} u \\
\dot{\bar{x}} &= \bar{A}\bar{x} + \bar{b}(u - z'p^*) \\
\dot{y} &= -\lambda y + \bar{c}\bar{x} + \bar{d}(u - z'p^*)
\end{aligned}$$

Monopoli combined this model with that of $\mathbb{M}_{\text{ref}}$ to obtain the *direct control model reference parameterization*

$$\dot{z} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} z + \begin{bmatrix} b \\ 0 \end{bmatrix} y + \begin{bmatrix} 0 \\ b \end{bmatrix} u \tag{2}$$

$$\dot{x} = \bar{A}x + \bar{b}(u - z'p^* - r) \tag{3}$$

$$\dot{\mathbf{e}}_{\mathbf{T}} = -\lambda \mathbf{e}_{\mathbf{T}} + \bar{c}x + \bar{d}(u - z'p^* - r) \tag{4}$$

Here $\mathbf{e}_{\mathbf{T}}$ is the *tracking error*

$$\mathbf{e}_{\mathbf{T}} \stackrel{\Delta}{=} y - y_{\text{ref}} \tag{5}$$

and $x \stackrel{\Delta}{=} \bar{x} - x_{\text{ref}}$. Note that it is possible to generate an asymptotically correct estimate $\widehat{z}$ of z using a copy of (2) with $\widehat{z}$ replacing z. To keep the exposition simple we're going to ignore the exponentially decaying estimation error $\widehat{z} - z$ and assume that z can be measured directly.

To solve the MRAC problem, Monopoli proposed a control law of the form

$$u = z'\widehat{p} + r \tag{6}$$

where $\widehat{p}$ is a suitably defined estimate of p^*. Motivation for this particular choice stems from the fact that if one knew p^* and were thus able to use the control $u = z'p^* + r$ instead of (6), then this would cause $\mathbf{e}_{\mathbf{T}}$ to tend to zero exponentially fast and tracking would therefore be achieved.

Monopoli proposed to generate $\widehat{p}$ using two sub-systems which we will refer to here as a "multi-estimator" and a " tuner" respectively. A *multi-estimator* $\mathbb{E}(\widehat{p})$ is a parameter-varying linear system with parameter $\widehat{p}$, whose inputs are u, y and r and whose output is an estimate $\widehat{e}$ of $\mathbf{e}_{\mathbf{T}}$ which would be asymptotically correct were $\widehat{p}$ held fixed at p^*. It turns out that there are two different but very similar types of multi-estimators which have the requisite properties. While Monopoli focused on just one, we will describe both since each is relevant to the present discussion. Both multi-estimators contain (2) as a sub-system.

2.1 Version 1

There are two versions of the adaptive controller which a relevant to the problem at hand. In this section we describe the multi-estimator and tuner which together with reference model (1) and control law (6), comprise the first version.

2.1.1 Multi-Estimator 1

The form of the first multi-estimator $\mathbb{E}_1(\widehat{p})$ is suggested by the readily verifiable fact that if H_1 and w_1 are $\bar{n} \times 2n$ and $\bar{n} \times 1$ signal matrices generated by the equations

$$\dot{H}_1 = \bar{A}H_1 + \bar{b}z' \qquad \text{and} \qquad \dot{w}_1 = \bar{A}w_1 + \bar{b}(u - r) \tag{7}$$

respectively, then $w_1 - H_1 p^*$ is a solution to (3). In other words $x = w_1 - H_1 p^* + \epsilon$ where ϵ is an initial condition dependent time function decaying to zero as fast as $e^{\bar{A}t}$. Again for simplicity we shall ignore ϵ. This means that (4) can be re-written as

$$\dot{\mathbf{e}}_{\mathbf{T}} = -\lambda \mathbf{e}_{\mathbf{T}} - (\bar{c}H_1 + \bar{d}z')p^* + \bar{c}w_1 + \bar{d}(u - r)$$

Thus a natural way to generate an estimate $\widehat{e}_1$ of $\mathbf{e}_{\mathbf{T}}$ is by means of the equation

$$\dot{\widehat{e}}_1 = -\lambda \widehat{e}_1 - (\bar{c}H_1 + \bar{d}z')\widehat{p} + \bar{c}w_1 + \bar{d}(u - r) \tag{8}$$

From this it clearly follows that the multi-estimator $\mathbb{E}_1(\widehat{p})$ defined by (2), (7) and (8) has the required property of delivering an asymptotically correct estimate $\widehat{e}_1$ of $\mathbf{e}_{\mathbf{T}}$ if $\widehat{p}$ is fixed at p^*.

2.1.2 Tuner 1

From (8) and the differential equation for $\mathbf{e}_{\mathbf{T}}$ directly above it, it can be seen that the estimation error[1]

$$e_1 \stackrel{\Delta}{=} \widehat{e}_1 - \mathbf{e}_{\mathbf{T}} \tag{9}$$

satisfies the error equation

$$\dot{e}_1 = -\lambda e_1 + \phi_1'(\widehat{p} - p^*) \tag{10}$$

where

$$\phi_1' = -(\bar{c}H_1 + \bar{d}z') \tag{11}$$

Prompted by this, Monopoli proposed to tune $\widehat{p}_1$ using the pseudo-gradient tuner

$$\dot{\widehat{p}}_1 = -\phi_1 e_1 \tag{12}$$

The motivation for considering this particular tuning law will become clear shortly, if it is not already.

2.1.3 What's Known About Version 1?

The overall model reference adaptive controller proposed by Monopoli thus consists of the reference model (1), the control law (6), the multi-estimator (2), (7), (8), the output estimation error (9) and the tuner (11), (12). *The open problem is to prove that this controller either solves the model reference adaptive control problem or that it does not.*

Much is known which is relevant to the problem. In the first place, note that (1), (2) together with (5) - (11) define a parameter varying linear system $\Sigma_1(\widehat{p})$ with input r, state $\{y_{\mathrm{ref}}, x_{\mathrm{ref}}, z, H_1, w_1, \widehat{e}_1, e_1\}$ and output e_1. The consequence of the assumption that every system in $\mathcal{M}$ is minimum phase is that $\Sigma_1(\widehat{p})$ is detectable through e_1 for every fixed value of $\widehat{p}$ [5]. Meanwhile the form of (10) enables one to show by direct calculation, that the rate of change of the partial Lyapunov function $V \stackrel{\Delta}{=} e_1^2 + ||\widehat{p} - p^*||^2$ along a solution to (12) and the equations defining $\Sigma_1(\widehat{p})$, satisfies

$$\dot{V} = -2\lambda e_1^2 \leq 0 \tag{13}$$

From this it is evident that V is a bounded monotone nonincreasing function and consequently that e_1 and $\widehat{p}$ are bounded wherever they exist. Using and the fact that $\Sigma_1(\widehat{p})$ is a *linear* parameter-varying system, it can be concluded that solutions exist globally and that e_1 and $\widehat{p}$ are bounded on

[1] Monopoli called e_1 an *augmented error.*

$[0, \infty)$. By integrating (13) it can also be concluded that e_1 has a finite $\mathcal{L}^2[0, \infty)$-norm and that $||e_1||^2 + ||\widehat{p} - p^*||^2$ tends to a finite limit as $t \to \infty$. Were it possible to deduce from these properties that $\widehat{p}$ tended to a limit $\bar{p}$, then it would possible to establish correctness of the overall adaptive controller using the detectability of $\Sigma_1(\bar{p})$.

There are two very special cases for which correctness has been established. The first is when the process models in $\mathcal{M}$ all have relative degree 1; that is when $\bar{n} = 1$. See the references cited in [3] for more on this special case. The second special case is when p^* is taken to be of the form $q^* k$ where k is a known vector and q^* is a scalar; in this case $\widehat{p} \stackrel{\Delta}{=} \widehat{q}k$ where $\widehat{q}$ is a scalar parameter tuned by the equation $\dot{\widehat{q}} = -k' \phi_1 e_1$ [6].

2.2 Version 2

In the sequel we describe the multi-estimator and tuner which together with reference model (1) and control law (6), comprise the second version of them adaptive controller relevant to the problem at hand.

2.2.1 Multi-Estimator 2

The second multi-estimator $\mathbb{E}_2(\widehat{p})$ which is relevant to the problem under consideration, is similar to $\mathbb{E}_1(\widehat{p})$ but has the slight advantage of leading to a tuner which is somewhat easier to analyze. To describe $\mathbb{E}_2(\widehat{p})$, we need first to define matrices

$$\bar{A}_2 \stackrel{\Delta}{=} \begin{bmatrix} \bar{A} & 0 \\ \bar{c} & -\lambda \end{bmatrix} \qquad \text{and} \qquad \bar{b}_2 \stackrel{\Delta}{=} \begin{bmatrix} \bar{b} \\ \bar{d} \end{bmatrix}$$

The form of $\mathbb{E}_2(\widehat{p})$ is motivated by the readily verifiable fact that if H_2 and w_2 are $(\bar{n}+1) \times 2n$ and $(\bar{n}+1) \times 1$ signal matrices generated by the equations

$$\dot{H}_2 = \bar{A}_2 H_2 + \bar{b}_2 z' \qquad \text{and} \qquad \dot{w}_2 = \bar{A}_2 w_2 + \bar{b}_2 (u - r) \tag{14}$$

then $w_2 - H_2 p^*$ is a solution to (3) - (4). In other words, $[\, x' \quad \mathbf{e_T} \,]' = w_2 - H_2 p^* + \epsilon$ where ϵ is an initial condition dependent time function decaying to zero as fast as $e^{\bar{A}_2 t}$. Again for simplicity we shall ignore ϵ. This means that

$$\mathbf{e_T} = \bar{c}_2 w_2 - \bar{c}_2 H_2 p^*$$

where $\bar{c}_2 = [\, 0 \quad \cdots \quad 0 \quad 1 \,]$. Thus in this case a natural way to generate an estimate $\widehat{e}_2$ of $\mathbf{e_T}$ is by means of the equation

$$\widehat{e}_2 = \bar{c}_2 w_2 - \bar{c}_2 H_2 \widehat{p} \tag{15}$$

It is clear that the multi-estimator $\mathbb{E}_2(\widehat{p})$ defined by (2), (14) and (15) has the required property of delivering an asymptotically correct estimate $\widehat{e}_2$ of $\mathbf{e_T}$ if $\widehat{p}$ is fixed at p^*.

2.2.2 Tuner 2

Note that in this case the estimation error

$$e_2 \stackrel{\Delta}{=} \widehat{e}_2 - \mathbf{e_T} \tag{16}$$

satisfies the error equation

$$e_2 = \phi_2' (\widehat{p}_2 - p^*) \tag{17}$$

where

$$\phi_2' = -\bar{c}_2 H_2 \tag{18}$$

Equation (17) suggests that one consider a pseudo-gradient tuner of the form

$$\dot{\widehat{p}} = -\phi_2 e_2 \tag{19}$$

2.2.3 What's Known About Version 2?

The overall model reference adaptive controller in this case, thus consists of the reference model (1), the control law (6), the multi-estimator (2), (14), (15), the output estimation error (16) and the tuner (18), (19). *The open problem is here to prove that this version of the controller either solves the model reference adaptive control problem or that it does not.*

Much is known about the problem. In the first place, (1), (2) together with (5), (6) (14) - (18) define a parameter varying linear system $\Sigma_2(\widehat{p})$ with input r, state $\{y_{\text{ref}}, x_{\text{ref}}, z, H_2, w_2\}$ and output e_2. The consequence of the assumption that every system in $\mathcal{M}$ is minimum phase is that this $\Sigma_2(\widehat{p})$ is detectable through e_2 for every fixed value of $\widehat{p}$ [5]. Meanwhile the form of (17) enables one to show by direct calculation, that the rate of change of the partial Lyapunov function $V \stackrel{\Delta}{=} ||\widehat{p} - p^*||^2$ along a solution to (19) and the equations defining $\Sigma_2(\widehat{p})$, satisfies

$$\dot{V} = -2\lambda e_2^2 \leq 0 \tag{20}$$

It is evident that V is a bounded monotone nonincreasing function and consequently that $\widehat{p}$ is bounded wherever they exist. From this and the fact that $\Sigma_2(\widehat{p})$ is a linear parameter-varying system, it can be concluded that solutions exist globally and that $\widehat{p}$ is bounded on $[0, \infty)$. By integrating (20) it can also be concluded that e_2 has a finite $\mathcal{L}^2[0, \infty)$-norm and that $||\widehat{p} - p^*||^2$ tends to a finite limit as $t \to \infty$. Were it possible to deduce from these properties that $\widehat{p}$ tended to a limit $\bar{p}$, then it would to establish correctness using the detectability of $\Sigma_2(\bar{p})$.

There is one very special cases for which correctness has been established [6]. This is when p^* is taken to be of the form q^*k where k is a known vector and q^* is a scalar; in this case $\widehat{p} \stackrel{\Delta}{=} \widehat{q}k$ where $\widehat{q}$ is a scalar parameter tuned by the equation $\dot{\widehat{q}} = -k'\phi_2 e_2$. The underlying reason why things go through is because in this special case, the fact that $||\widehat{p} - p^*||^2$ and consequently $||\widehat{q} - q^*||$ tend to a finite limits, means that $\widehat{q}$ tends to a finite limit as well.

3 The Essence of the Problem

In this section we write down a stripped down version of the problem which retains all the essential feature which need to be overcome in order to decide whether or not Monopoli's controller is correct. We do this only for version 2 of the problem and only for the case when $r = 0$ and $\bar{n} = 1$. Thus in this case we can take $\bar{A}_2 = -\lambda$ and $\bar{b}_2 = 1$. Assuming the reference model is initialized at 0, dropping the subscript 2 throughout, and writing ϕ' for $-H$, the system to be analyzed reduces to

$$\begin{aligned}
\dot{z} &= \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} z + \begin{bmatrix} b \\ 0 \end{bmatrix} (w + \phi' p^*) + \begin{bmatrix} 0 \\ b \end{bmatrix} \widehat{p}' z && (21) \\
\dot{\phi} &= -\lambda \phi - z && (22) \\
\dot{w} &= -\lambda w + \widehat{p}' z && (23) \\
e &= \phi'(\widehat{p} - p^*) && (24) \\
\dot{\widehat{p}} &= -\phi e && (25)
\end{aligned}$$

To recap, p^* is unknown and constant but is such that the linear parameter-varying system $\Sigma(\widehat{p})$ defined by (21) to (24) is detectable through e for each fixed value of $\widehat{p}$. Solutions to the system (21) - (25) exist globally. The parameter vector $\widehat{p}$ and integral square of e are bounded on $[0, \infty)$ and $||\widehat{p} - p^*||$ tends to a finite limit as $t \rightarrow \infty$. *The open problem here is to show for every initialization of (21)-(25), that the state of* $\Sigma(\widehat{p})$ *tends to* 0 *or that it does not.*

References

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