Input-Output Gains of Switched Linear Systems* †

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The problems which follow are motivated primarily by a desire to develop a bona fide performance-based theory of adaptive control. At this point it seems fairly clear that such a theory will require a much better understanding of certain types of linear time-varying systems than we have at present. In an adaptive context, switched (as opposed to continuously tuned) linear systems seem to be the most tractable and it is for this reason that the problems formulated here are for switched linear systems.

Let \( P \) be either a finite or compact subset of a real, finite dimensional linear space and let \( M_p : P \to \mathbb{R}^{n \times n}, D_p : P \to \mathbb{R}^{n \times n_u}, \) and \( H_p : P \to \mathbb{R}^{n_y \times n} \) be given continuous functions. Assume that for each \( p \in P \), \( M_p \) is a stability matrix. Let \( \tau_D \) be a given positive number. Let \( S \) denote the set of all piecewise-constant signals \( \sigma : [0, \infty) \to P \) such that either (a) \( \sigma \) switches at most once or (b) \( \sigma \) switches more than once and the time difference between each two successive switching times is bounded below by \( \tau_D \).

For each \( \sigma \) in \( S \), the preceding defines a time-varying linear system of the form

\[
\Sigma_\sigma \triangleq \begin{cases} 
\dot{x} &= M_\sigma x + D_\sigma u \\
y &= H_\sigma x 
\end{cases}
\]

where \( u \) is an integrable input signal taking values in \( \mathbb{R}^{n_u} \). Thus if \( x(0) \overset{\Delta}{=} 0 \), then \( y = Y_\sigma(u) \), where \( Y_\sigma \) is the input-output mapping

\[
u \mapsto \int_0^t H_\sigma(t) \Phi_\sigma(t, \tau) D_\sigma(\tau) d\tau,
\]

and \( \Phi_\sigma \) is the state transition matrix of \( M_\sigma \). Let prime denote transpose and, for any integrable, vector-valued signal \( v \) on \( [0, \infty) \), let \( \| \cdot \| \) denote the two-norm

\[
\| v \| \overset{\Delta}{=} \sqrt{\int_0^\infty v'(t) v(t) dt}
\]

Write \( L_2 \) for the space of all signals with finite two-norms. The input-output gain of \( \Sigma_\sigma \) is then the induced two-norm

\[
\mu_\sigma \overset{\Delta}{=} \inf \{ \gamma : \| Y_\sigma(u) \| \leq \gamma \| u \|, \ \forall u \in L_2 \}
\]

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Define the gain $\mu$ of the multi-system $\{(H_p, M_p, D_p), p \in \mathcal{P}\}$ to be

$$\mu \triangleq \sup_{\sigma \in \mathcal{S}} \mu_{\sigma}.$$  

**Main Problem:** Derive conditions in terms of $\tau_D$ and the multi-system $\{(H_p, M_p, D_p), p \in \mathcal{P}\}$ under which $\mu$ is a finite number. Assuming these conditions hold, characterize $\mu$ in terms of $\{(H_p, M_p, D_p), p \in \mathcal{P}\}$ and $\tau_D$.

The problem just posed implicitly contains as a sub-problem the question of whether or not the time varying matrix $M_\sigma$ is exponentially stable for every $\sigma \in \mathcal{S}$. This sub-problem, in turn, is arguably the most important open problem in the theory of switched linear systems. Now it is well-known that $M_\sigma$ will be exponentially stable for each $\sigma \in \mathcal{S}$ if either $\tau_D$ is sufficiently large or if the diameter of $\mathcal{P}$ is sufficiently small [1, 2]. Some recent but special results on switched linear systems can be found in [3, 4, 5, 6, 7, 8] and the references therein. In addition, there is an extensive literature on the stability of time varying matrices (see for example pages 116–119 of [9], pages 239–250 of [10], the important survey paper [11] and the “preliminaries” of [12]) but these results are not specific to switched linear systems. Apart from these, very little of a general nature is known about the stability of $M_\sigma$, to say nothing of how to characterize $\mu$. It is of course quite easy to calculate stability bounds for $\tau_D$ and bounds for $\mu$ in terms of normed values of the multi-system’s matrices, but these bounds are invariably state-coordinate dependent. They ought not be.

As stated, the main problem is extremely challenging. Moreover while the problem is formulated in the state space, $\mu$ and the stability of $M_\sigma$ are invariant under state coordinate changes of the form $\{(H_p, M_p, D_p), p \in \mathcal{P}\} \rightarrow \{(H_p T^{-1}, T M_p T^{-1}, T D_p), p \in \mathcal{P}\}$ where $T$ is a constant nonsingular matrix. One would thus like to see the main problem reformulated in a way which takes this invariance into account. We will now formulate a simpler problem which accomplishes this.

Let us consider the case when for $p \in \mathcal{P}$, $H_p$ and $D_p$ are row and column vectors $h_p$ and $d$ respectively, with $d$ not depending on $p$. Let us restrict $M_p$ to be of the form $M_p \triangleq A + c_p b f_p$ where $A$ is a constant matrix, $b$ is a constant $n$-vector and for $p \in \mathcal{P}$, $c_p$ and $f_p$ are row vectors such that $A + c_p b f_p$ is exponentially stable. Thus we are restricting attention to a multi-system of the form $\{(h_p, A + c_p b f_p, d), p \in \mathcal{P}\}$. Multi-systems such as this arise quite commonly in adaptive control [2]. For simplicity, suppose that $\mathcal{P}$ is the finite set $\mathcal{P} \triangleq \{1,2,\ldots,m\}$. Suppose, in addition, without loss of generality, that $(A, [b \ d])$ is controllable and that $(C, A)$ is observable, where

$$C = [f_1' \ c_1 \ h_1' \ f_2' \ c_2 \ h_2' \ \cdots \ f_m' \ c_m \ h_m']'$$

Under these conditions the multi-system $\{(h_p, A + c_p b f_p, d), p \in \mathcal{P}\}$ uniquely determine a $3m \times 2$ strictly proper rational matrix $W(s) \triangleq C(sI - A)^{-1} [b \ d]$ of McMillan Degree $n$. In the light of classical realization theory, it is easy to see that by reversing steps, $W(s)$ uniquely determines $\{(h_p, A + c_p b f_p, d), p \in \mathcal{P}\}$ up to a similarity transformation of the form

$$\{(h_p, A + c_p b f_p, d), p \in \mathcal{P}\} \leftrightarrow \{(h_p T^{-1}, T(A + c_p b f_p) T^{-1}, T d), p \in \mathcal{P}\}$$

Since neither the stability of $A + c_\sigma b f_\sigma$ nor the multi-system gain $\mu$ depend on $T$, we can meaningfully formulate the following problem.

**Simplified Problem:** Derive conditions in terms of $\tau_D$ and the multi-system rational matrix $W(s)$ under which $\mu$ is a finite number. Assuming these conditions hold, characterize $\mu$ in terms of $W(s)$ and $\tau_D$. 

2
Without destroying its significance, the preceding problem can be simplified further by restricting \( \sigma \) to be in the class of “clocked” switching signals \( S_C \) consisting of those switching signals in \( S \) for which the switching times are elements of the infinite sequence \( 0, \tau_D, 2\tau_D, 3\tau_D, \ldots \). One might also try to determine the limiting behavior of \( \mu \) as \( \tau_D \to \infty \). It is for example known that in the limit, \( \mu \) equals the maximum over \( \mathcal{P} \) of the \( H^\infty \) norm of the transfer function \( h_p(sI - A - dc_p - bf_p)^{-1}d \) [13].

References


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1Although further restricting \( \sigma \) to be periodic would make things much more tractable, it would at the same time destroy completely the significance of the problem to adaptive control.