Input-Output Gains of Switched Linear Systems* †

J. P. Hespanha and A. S. Morse
Center for Computational Vision and Control
Yale University, New Haven, CT 06520

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The problems which follow are motivated primarily by a desire to develop a bona fide performance-based theory of adaptive control. At this point it seems fairly clear that such a theory will require a much better understanding of certain types of linear time-varying systems than we have at present. In an adaptive context, switched {as opposed to continuously tuned} linear systems seem to be the most tractable and it is for this reason that the problems formulated here are for switched linear systems.

Let $\mathcal{P}$ be either a finite or compact subset of a real, finite dimensional linear space and let $M_p : \mathcal{P} \to \mathbb{R}^{n\times n}$, $D_p : \mathcal{P} \to \mathbb{R}^{n\times n_u}$, and $H_p : \mathcal{P} \to \mathbb{R}^{n_y\times n}$ be given continuous functions. Assume that for each $p \in \mathcal{P}$, M_p is a stability matrix. Let τ_D be a given positive number. Let $\mathcal{S}$ denote the set of all piecewise-constant signals $\sigma : [0,\infty) \to \mathcal{P}$ such that either (a) σ switches at most once or (b) σ switches more than once and the time difference between each two successive switching times is bounded below by τ_D.

For each σ in $\mathcal{S}$, the preceding defines a time-varying linear system of the form

$$\Sigma_\sigma \triangleq \left\{ \begin{array}{rcl} \dot{x} & = & M_\sigma x + D_\sigma u \\ y & = & H_\sigma x \end{array} \right\}$$

where u is an integrable input signal taking values in $\mathbb{R}^{n_u}$. Thus if $x(0) \triangleq 0$, then $y = Y_\sigma(u)$, where Y_σ is the input-output mapping

$$u \longmapsto \int_0^t H_{\sigma(t)}\Phi_\sigma(t,\tau)D_{\sigma(\tau)}d\tau,$$

and Φ_σ is the state transition matrix of M_σ. Let prime denote transpose and, for any integrable, vector-valued signal v on $[0,\infty)$, let $||\cdot||$ denote the two-norm

$$||v|| \triangleq \sqrt{\int_0^\infty v'(t)v(t)dt}$$

Write $\mathcal{L}_2$ for the space of all signals with finite two-norms. The input-output gain of Σ_σ is then the induced two-norm

$$\mu_\sigma \triangleq \inf\{\gamma : ||Y_\sigma(u)|| \le \gamma||u||,\ \forall u \in \mathcal{L}_2\}$$

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Define the gain μ of the *multi-system* $\{(H_p, M_p, D_p), p \in \mathcal{P}\}$ to be

$$\mu \triangleq \sup_{\sigma \in \mathcal{S}} \mu_\sigma$$

Main Problem: *Derive conditions in terms of τ_D and the multi-system $\{(H_p, M_p, D_p), p \in \mathcal{P}\}$ under which μ is a finite number. Assuming these conditions hold, characterize μ in terms of $\{(H_p, M_p, D_p), p \in \mathcal{P}\}$ and τ_D.*

The problem just posed implicitly contains as a sub-problem the question of whether or not the time varying matrix M_σ is exponentially stable for every $\sigma \in \mathcal{S}$. This sub-problem, in turn, is arguably the most important open problem in the theory of switched linear systems. Now it is well-known that M_σ will be exponentially stable for each $\sigma \in \mathcal{S}$ if either τ_D is sufficiently large or if the diameter of $\mathcal{P}$ is sufficiently small [1, 2]. Some recent but special results on switched linear systems can be found in [3, 4, 5, 6, 7, 8] and the references therein. In addition, there is an extensive literature on the stability of time varying matrices {see for example pages 116–119 of [9], pages 239–250 of [10], the important survey paper [11] and the "preliminaries" of [12]} but these results are not specific to switched linear systems. Apart from these, very little of a general nature is known about the stability of M_σ, to say nothing of how to characterize μ. It is of course quite easy to calculate stability bounds for τ_D and bounds for μ in terms of normed values of the multi-system's matrices, but these bounds are invariably state-coordinate dependent. They ought not be.

As stated, the main problem is extremely challenging. Moreover while the problem is formulated in the state space, μ and the stability of M_σ are invariant under state coordinate changes of the form $\{(H_p, M_p, D_p), p \in \mathcal{P}\} \longmapsto \{(H_pT^{-1}, TM_pT^{-1}, TD_p), p \in \mathcal{P}\}$ where T is a constant nonsingular matrix. One would thus like to see the main problem reformulated in a way which takes this invariance into account. We will now formulate a simpler problem which accomplishes this.

Let us consider the case when for $p \in \mathcal{P}$, H_p and D_p are row and column vectors h_p and d respectively, with d not depending on p. Let us restrict M_p to be of the form $M_p \triangleq A + dc_p + bf_p$ where A is a constant matrix, b is a constant n-vector and for $p \in \mathcal{P}$, c_p and f_p are row vectors such that $A + dc_p + bf_p$ is exponentially stable. Thus we are restricting attention to a multi-system of the form $\{(h_p, A + dc_p + bf_p, d), p \in \mathcal{P}\}$. Multi-systems such as this arise quite commonly in adaptive control [2]. For simplicity, suppose that $\mathcal{P}$ is the finite set $\mathcal{P} \triangleq \{1, 2, \ldots, m\}$. Suppose, in addition, without loss of generality, that $(A, [\,b \quad d\,])$ is controllable and that (C, A) is observable, where

$$C = [\, f_1' \quad c_1' \quad h_1' \quad f_2' \quad c_2' \quad h_2' \quad \cdots \quad f_m' \quad c_m' \quad h_m' \,]'$$

Under these conditions the multi-system $\{(h_p, A + dc_p + bf_p, d), p \in \mathcal{P}\}$ uniquely determine a $3m \times 2$ strictly proper rational matrix $W(s) \triangleq C(sI - A)^{-1}[\,b \quad d\,]$ of McMillan Degree n. In the light of classical realization theory, it is easy to see that by reversing steps, $W(s)$ uniquely determines $\{(h_p, A + dc_p + bf_p, d), p \in \mathcal{P}\}$ up to a similarity transformation of the form

$$\{(h_p, A + dc_p + bf_p, d),\ p \in \mathcal{P}\} \longmapsto \{(h_pT^{-1}, T(A + dc_p + bf_p)T^{-1}, Td),\ p \in \mathcal{P}\}$$

Since neither the stability of $A + dc_\sigma + bf_\sigma$ nor the multi-system gain μ depend on T, we can meaningfully formulate the following problem.

Simplified Problem: *Derive conditions in terms of τ_D and the multi-system rational matrix $W(s)$ under which μ is a finite number. Assuming these conditions hold, characterize μ in terms of $W(s)$ and τ_D.*

Without destroying its significance, the preceding problem can be simplified further by restricting σ to be in the class of "clocked" switching signals $\mathcal{S}_C$ consisting of those switching signals in $\mathcal{S}$ for which the switching times are elements of the infinite sequence $0,\ \tau_D,\ 2\tau_D,\ 3\tau_D, \ldots$[1] One might also try to determine the limiting behavior of μ as $\tau_D \to \infty$. It is for example known that in the limit, μ equals the maximum over $\mathcal{P}$ of the $\mathcal{H}^\infty$ norm of the transfer function $h_p(sI - A - dc_p - bf_p)^{-1}d$ [13].

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[1] Although further restricting σ to be periodic would make things much more tractable, it would at the same time destroy completely the significance of the problem to adaptive control.