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Stabilization of nonholonomic integrators via logic-based switching[1]

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A hybrid control law employing switching and logic is proposed to stabilize a "nonholonomic integrator". Results concerning asymptotic stability and exponential convergence to the origin are derived.

Abstract

This paper explains how to stabilize a "nonholonomic integrator" using a hybrid control law employing switching and logic. Results concerning asymptotic stability and exponential convergence to the origin are derived. The notion of a (positively) invariant set is extended to hybrid systems and sufficient conditions for invariance are presented. The verification of these conditions does not require the computation of the state trajectories and their use goes beyond the analysis of the system presented in this paper.

Keywords: Logic-based switching; Nonholonomic systems; Hybrid systems; Nonlinear control

1. Introduction

Over the last decade there has been a great deal of research concerned with the problem of stabilizing systems that are locally null controllable but fail to meet Brockett's (1983) condition for smooth stabilizability: *Given the system*

$$\dot{x} = f(x, u), \qquad x(t_0) = x_0, \qquad f(0, 0) = 0, \tag{1}$$

with $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *continuously differentiable. If Eq. (1) is smoothly stabilizable, i.e. there exists a continuously differentiable function* $g: \mathbb{R}^n \to \mathbb{R}^m$ *such that the origin is an asymptotically stable equilibrium point of* $\dot{x} = f(x, g(x))$, *with stability defined in the Lyapunov sense, then the image of f must contain an open neighborhood of the origin.*

Sontag 1990 noted that this condition extends to the class of time-invariant feedback laws that are only locally Lipschitz and recently it was shown (Ryan, 1994; Coron et al., 1995) that Brockett's condition also extends to an even larger class that includes a wide variety of time-invariant discontinuous feedback laws, when one demands that the origin be an asymptotically stable equilibrium point for all Filippov's (1964) solutions of the closed-loop system. A prototype example of a system that is not smoothly stabilizable is the so called "nonholonomic integrator" (Brockett, 1983):

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = u_2, \quad \dot{x}_3 = x_1 u_2 - x_2 u_1, \tag{2}$$

where $x \triangleq [x_1\ x_2\ x_3]' \in \mathbb{R}^3$ and $u \triangleq [u_1\ u_2]' \in \mathbb{R}^2$. Since the image of the map $[x'\ u']' \mapsto [u_1\ u_2\ x_1u_2 - x_2u_1]'$ does not contain the point $[0\ 0\ \varepsilon]'$ for any $\varepsilon \neq 0$, Brockett's condition implies that there is no time-invariant continuously differentiable control law that asymptotically stabilizes the origin. It turns out that any kinematic completely nonholonomic system with three states and two control inputs can be converted to

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a nonholonomic integrator by a local coordinate transformation (Murray and Sastry, 1993).

The difficulties implied by Brockett's condition can be avoided using time-varying periodic controllers, stochastic control laws, and sliding modes control laws. The control proposed in this paper falls into the class of hybrid control laws, namely those employing both continuous dynamics and discrete logic . Applications of this type of laws to nonholonomic systems can be found in Bloch et al. (1992), Kolmanovsky et al. (1994), Hespanha (1996), Pait and Piccoli (1996), and Kolmanovsky and McClamroch (1996). In the first two references global convergence to the origin is achieved in finite time; however, these controls may result in chattering in the presence of unmodeled dynamics. Kolmanovsky and McClamroch (1996) propose a time-varying hybrid controller to asymptotically stabilize a general class of nonholonomic systems represented in power form. The reader is referred to Kolmanovsky and McClamroch (1995) for an extensive survey of recent results concerned with the control of nonholonomic systems. The present paper continues a line of research started by Hespanha (1996), where a time-invariant hybrid control law that achieves polynomial convergence to the origin was proposed. In this paper we present a time-invariant hybrid control law that guarantees global asymptotic stability with exponential convergence to the origin of the state of the nonholonomic integrator. Exponential stabilization of systems like the nonholonomic integrator was also achieved by M'Closkey and Murray (1997) using nonsmooth, continuous, time-varying control laws.

The analysis of the closed-loop hybrid system in this paper uses the notion of invariant sets extended to hybrid systems. Informally, a subset $\mathscr{A}$ of the state space of a hybrid system Σ is said to be invariant if for any initialization of the state of Σ within $\mathscr{A}$, the corresponding trajectory remains in $\mathscr{A}$ for all future times. The definition is an obvious generalization of the concept of positively invariant set for systems only with continuous dynamics. This paper introduces tests to determine if a given set is invariant for a certain hybrid system. These tests do not require the computation of the hybrid system's state trajectory and their use goes beyond the present application. The use of invariance in a hybrid systems context was touched upon by Branicky (1995).

The remaining of the paper is organized as follows. In Section 2 the proposed hybrid control law is presented and briefly motivated. The main result of the paper is stated in Section 3, namely that the control law described in Section 2 makes the closed-loop hybrid system asymptotically stable with exponential convergence of the continuous part of the state to the origin. In Section 4 the concept of invariant set for an hybrid system is introduced and basic tests for invariance of sets are presented. In Section 5 these tests are used to prove the stability of the closed-loop system. Finally, Section 6 contains a brief discussion of the results achieved so far and some directions for future research.

2. Switching controller

Consider again the nonholonomic integrator

$$\dot{x}_1 = u_1, \qquad \dot{x}_2 = u_2, \qquad \dot{x}_3 = x_1 u_2 - x_2 u_1.$$

No matter what control law is used, whenever x_1 and x_2 are both zero, $\dot{x}_3$ will also be zero and x_3 will remain constant. Furthermore, whenever x_1 and x_2 are "small", only "large" control signals will be able to produce significant changes in x_3. A plausible strategy to make the origin an attractor of the close-loop system is to keep the state away from the axes $x_1 = x_2 = 0$ while x_3 is large and, as x_3 decreases, to let x_1 and x_2 became small. Several control laws can achieve the aforementioned type of behavior. The one presented in this paper has the virtue of being easy to analyze, not only in terms of stability, but also in terms of speed of convergence. The control law proposed is constructed as follows:

1. Pick four continuous, monotone nondecreasing, functions $\pi_j : [0, +\infty) \to \mathbb{R}$, $j \in \mathscr{S} \triangleq \{1, 2, 3, 4\}$, with the following properties:

(i) $\pi_j(0) = 0$ for each $j \in \mathscr{S}$, and $0 < \pi_1(w) < \pi_2(w) < \pi_3(w) < \pi_4(w)$ for every $w > 0$.
(ii) π_1 and π_2 are bounded.
(iii) π_1 is such that if $w \to 0$ exponentially fast[2] then $w/\pi_1(w) \to 0$ exponentially fast.
(iv) π_4 is smooth on some nonempty interval $(0, c]$, and

$$\pi_4'(w) < \frac{\pi_4(w)}{w}, \quad w \in (0, c]. \tag{3}$$

Moreover, if $w \to 0$ exponentially fast then $\pi_4(w) \to 0$ exponentially fast.

2. Partition $\mathbb{R}^3$ into 4 overlapping regions

$$\mathscr{R}_1 \triangleq \{x \in \mathbb{R}^3 : 0 \le x_1^2 + x_2^2 < \pi_2(x_3^2)\},$$

$$\mathscr{R}_2 \triangleq \{x \in \mathbb{R}^3 : \pi_1(x_3^2) < x_1^2 + x_2^2 < \pi_4(x_3^2)\},$$

$$\mathscr{R}_3 \triangleq \{x \in \mathbb{R}^3 : \pi_3(x_3^2) < x_1^2 + x_2^2\},$$

$$\mathscr{R}_4 \triangleq \{0\}.$$

3. Define the control law

$$u = g_\sigma(x), \quad t \ge t_0, \tag{4}$$

[2] A signal $x : [0, \infty) \to \mathbb{R}^n$ is said to converge to zero exponentially fast if there exist positive constants a, λ such that $\|x(t)\| \le a e^{-\lambda t}$ for every $t \ge 0$.

where σ is a piecewise constant, continuous from the right at every point, switching signal taking values on $\mathscr{S}$ and, for each $x \in \mathbb{R}^3$,

$$g_1(x) \triangleq \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad g_2(x) \triangleq \begin{bmatrix} x_1 + \dfrac{x_2 x_3}{x_1^2 + x_2^2} \\ x_2 - \dfrac{x_1 x_3}{x_1^2 + x_2^2} \end{bmatrix}, \tag{5}$$

$$g_3(x) \triangleq \begin{bmatrix} -x_1 + \dfrac{x_2 x_3}{x_1^2 + x_2^2} \\ -x_2 - \dfrac{x_1 x_3}{x_1^2 + x_2^2} \end{bmatrix}, \qquad g_4(x) \triangleq \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{6}$$

The signal σ is determined recursively by

$$\sigma = \phi(x, \sigma^-), \quad t \geq t_0, \tag{7}$$

where, for each $t > t_0$, $\sigma^-(t)$ denotes the limit from the left of $\sigma(\tau)$ as $\tau \uparrow t$, $\sigma^-(t_0)$ is equal to some element of $\mathscr{S}$ that effectively initializes (7), and $\phi : \mathbb{R}^3 \times \mathscr{S} \to \mathscr{S}$ is the transition function defined by

$$\phi(x, j) = \begin{cases} j & \text{if } x \in \mathscr{R}_j, \\ \max\{i \in \mathscr{S} : x \in \mathscr{R}_i\} & \text{if } x \notin \mathscr{R}_j, \end{cases} \qquad x \in \mathbb{R}^3,\ j \in \mathscr{S}. \tag{8}$$

Example 1. A typical choice for the functions π_j is $\pi_1(w) = (1 - e^{-\sqrt{w}})$, $\pi_2 = 2\pi_1$, $\pi_3 = 3\pi_1$, and $\pi_4 = 4\pi_1$. Fig. 1 shows the projection of the corresponding regions $\mathscr{R}_1$, $\mathscr{R}_2$, and $\mathscr{R}_3$ into the $(x_3^2, x_1^2 + x_2^2)$-space.

The type of control proposed is similar to that of Back et al. (1993) and can be viewed as an extension of the hysteresis switching algorithm considered by Morse et al. (1992). Its appeal comes from the fact that it naturally excludes the possibility of infinitely fast chattering and therefore does not require the concept of generalized solution in Filippov's sense (Guldner and Utkin, 1994; Bloch and Drakunov, 1996).

3. Main result

The aim of this section is to study the closed-loop hybrid dynamical system described in the previous section. The relevant equations are

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = u_2, \quad \dot{x}_3 = x_1 u_2 - x_2 u_1,$$

$$u = g_\sigma(x), \quad \sigma = \phi(x, \sigma^-),$$

where $x \triangleq [x_1\ x_2\ x_3]' \in \mathbb{R}^3$, $u \triangleq [u_1\ u_2]' \in \mathbb{R}^2$, and $g_\sigma(x)$ and $\phi(x, \sigma)$ are defined by Eqs. (5)–(6) and (8), respectively. Although the closed-loop system is not globally Lipschitz, global existence of solutions can be easily justified. Indeed, defining $w_1 \triangleq x_3^2$ and $w_2 \triangleq x_1^2 + x_2^2$, simple algebra shows that

$$\dot{w}_1 \leq 2w_1 + w_2, \quad \dot{w}_2 \leq 2w_2 + 2.$$

Since the bounds for the right-hand sides of the above equations are globally Lipschitz with respect to w_1 and w_2, these variables and their derivatives must be bounded on any finite interval. Moreover, the distance between two points in the (w_1, w_2) -space where consecutive switchings can occur is always nonzero. The boundedness of $\dot{w}_1$ and $\dot{w}_2$ thus guarantees that the time interval between consecutive discontinuities of σ is always positive, i.e., σ is piecewise constant. Now the system of differential equations given by Eqs. (2) and (4) can be written as

$$\dot{x} = f_{\sigma(t)}(x), \tag{9}$$

with each f_j, $j \in \mathscr{S}$, locally Lipschitz. Since it has been established that σ is piecewise constant, the right-hand side of Eq. (9) is locally Lipschitz with respect to x and piecewise continuous with respect to t. This together with the fact that x is bounded on any finite interval (because the same is true for w_1 and w_2) guarantees that the solution exists globally and is unique. The fact that the regions $\mathscr{R}_i$, $i \in \mathscr{S}$ used to define the transition function ϕ are open subsets of $\mathbb{R}^3$ guarantees that σ is indeed continuous from the right at every point.

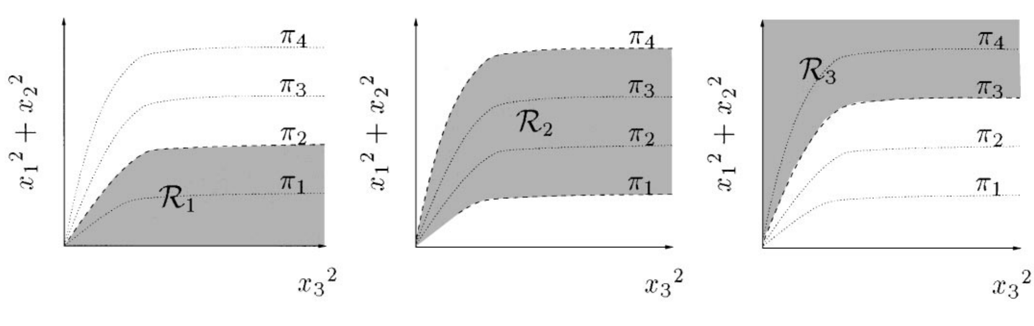

Fig. 1. Projection of the regions $\mathscr{R}_1$, $\mathscr{R}_2$, and $\mathscr{R}_3$ into the $(x_3^2, x_1^2 + x_2^2)$ -space.

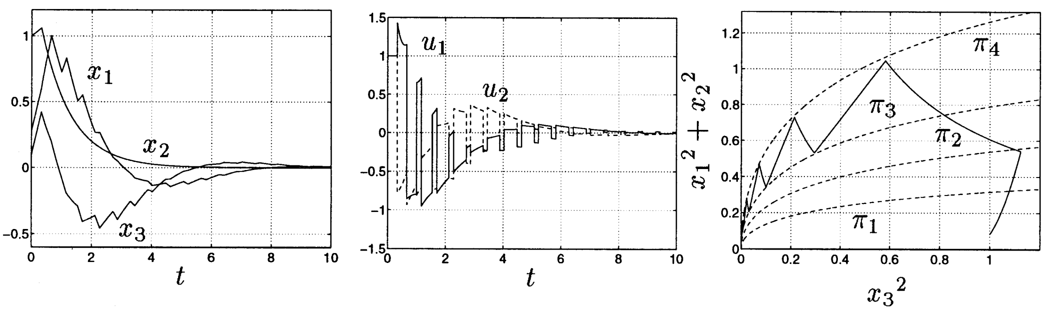

Fig. 2. Simulation of the closed-loop hybrid system Σ: x versus time, u versus time, and projection of x into the $(x_3^2, x_1^2 + x_2^2)$ -space.

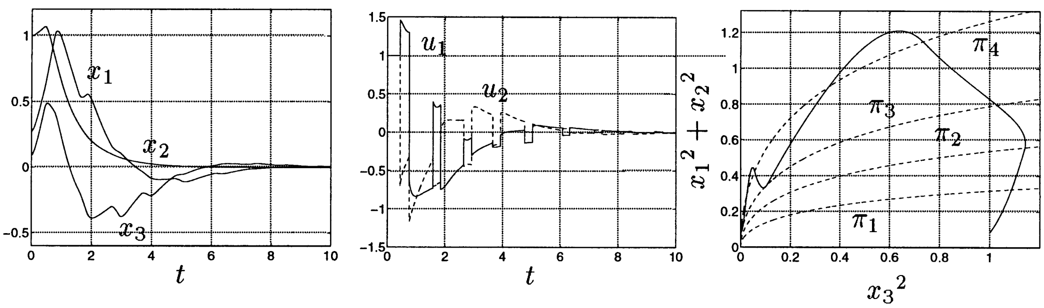

Fig. 3. Simulation of the closed-loop hybrid system with modeling errors: x versus time, u versus time, and projection of x into the $(x_3^2, x_1^2 + x_2^2)$ -space.

The above argument excludes the possibility of infinitely fast chattering in the sense that the interval between consecutive discontinuities of σ is bounded below by a positive constant on any finite interval. Latter we are going to see that the interval between consecutive switchings is bounded away from zero even as time goes to infinity (cf. Remark 5 in Section 5).

The usual definition of Lyapunov stability extends in a natural way to hybrid systems: the origin is a *Lyapunov stable equilibrium point* of the hybrid system Σ defined by $\dot{x} = f_\sigma(x)$, $\sigma = \phi(x, \sigma^-)$, $x \in \mathscr{X} \triangleq \mathbb{R}^n$, $\sigma \in \mathscr{S}$ if

(i) $f_j(0) = 0$ for each $j \in \mathscr{S}$ such that $\phi(0, j) = j$, and
(ii) for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $x_0 \in \mathscr{X}$ and every $\sigma_0 \in \mathscr{S}$ with $\|x_0\| < \delta$, any solution $\{x, \sigma\}$ to Σ with $x(t_0) = x_0$ and $\sigma^-(t_0) = \sigma_0$, exists globally and $\|x(t)\| < \varepsilon$ for $t \geq t_0$.

Moreover, if for any initial conditions, the continuous part of the state x converges to the origin, then the origin is said to be *globally asymptotically stable*. The main result of this paper is the following theorem.

Theorem 2. *Let Σ denote the hybrid system defined by Eqs.* (2), (4), *and* (7).
(1) *The origin is a globally asymptotically stable equilibrium point of Σ.*
(2) *The continuous part x of the state of Σ and the control signal u converge to zero exponentially fast along any solution to Σ.*

Fig. 2 shows a simulation[3] of the closed-loop system Σ defined by Eqs. (2), (4), and (7), with

$$\pi_1(w) = 0.5(1 - e^{-\sqrt{w}}), \quad \pi_2(w) = 1.7\pi_1(w), \tag{10}$$

$$\pi_3(w) = 2.5\pi_1(w), \qquad \pi_4(w) = 4\pi_1(w), \quad w \geq 0. \tag{11}$$

As expected, x and u converge to zero exponentially fast. For the trajectory shown in these plots, not only is chattering precluded on any finite time interval, but it is also true that the interval between consecutive switchings is bounded away from zero as time goes to infinity. It turns out that this is true for *any* trajectory of this hybrid system (cf. Remark 5 in Section 5).

To test the robustness of the controller proposed with respect to modeling errors, the hybrid controller defined by Eqs. (4) and 7 was also applied to the system

$$\dot{x}_1 = v_1, \quad \dot{x}_2 = v_2, \quad \dot{x}_3 = x_1 v_2 - x_2 v_1, \tag{12}$$

$$\dot{v}_1 = -10v_1 + 9.5u_1, \quad \dot{v}_2 = -10v_2 + 10.5u_2. \tag{13}$$

This system consists of a nonholonomic integrator (12) in cascade with first-order low-pass filters (13) with DC gains close but not equal to 1. Eqs. (13) could model, for example, simple actuator dynamics. Fig. 3 shows a

[3] The reader wishing to experiment with these simulations can obtain the MATLAB/SIMULINK files at the website http: //cvc.yale.edu.

simulation of the closed-loop hybrid system defined by Eqs. (12) and (13), (4), and (7). In this simulation one can see that the "actuator dynamics" (13) do not compromise the exponential convergence to the origin nor do they introduce chattering.

4. Invariant sets

To prove Theorem 2 we need to extend the notion of an invariant set to hybrid systems. To this effect consider a hybrid system Σ defined by the ordinary differential equation

$$\dot{x} = f_\sigma(x), \quad t \geq t_0, \tag{14}$$

together with the recursive equation

$$\sigma = \phi(x, \sigma^-), \quad t \geq t_0, \tag{15}$$

where $x \in \mathscr{X} \triangleq \mathbb{R}^n$, $\sigma \in \mathscr{S}$, and each $f_j : \mathscr{X} \to \mathscr{X}$, $j \in \mathscr{S}$, is a Lipschitz continuous function. A pair of sets $\{\mathscr{Z}, \mathscr{J}\}$ with $\mathscr{Z} \subset \mathscr{X}$ and $\mathscr{J} \subset \mathscr{S}$ is *invariant with respect to* Σ if, for every $x_0 \in \mathscr{Z}$ and every $\sigma_0 \in \mathscr{S}$, any solution $\{x, \sigma\}$ to Σ with $x(t_0) = x_0$ and $\sigma^-(t_0) = \sigma_0$ remains in $\mathscr{Z} \times \mathscr{J}$ for all times $t \geq t_0$ for which the solution is defined. In this paper we restrict our attention to systems for which σ is continuous from the right at every point. In Section 3 it has already been established that this happens for the hybrid system considered here.

The following lemma provides a procedure to prove invariance of a given pair of sets by observing the values of the functions $f_j : \mathscr{X} \to \mathscr{X}$, $j \in \mathscr{S}$ at the boundary of $\mathscr{Z}$. The following terminology is used: Given a subset $\mathscr{Z}$ of $\mathscr{X}$, a vector[4] $\mathbf{v} = (x, v) \in \mathscr{X} \times \mathscr{X}$ at a point x on the boundary of $\mathscr{Z}$ is said to *point towards* $\mathscr{Z}$ if there exist positive constants h and r such that the cone with spherical base $C[\mathbf{v}, h, r]$ shown in Fig. 4 and defined by

$$C[\mathbf{v}, h, r] \triangleq \{z \in \mathscr{X} : \| x + \rho h v - z \| \leq \rho r, \quad \rho \in [0, 1]\}$$

is contained in $\mathscr{Z}$. Here we are using the norm topology on $\mathscr{X} \triangleq \mathbb{R}^n$.

Lemma 3. *Consider the hybrid system* Σ *defined by Eqs.* (14) *and* (15) *and a pair of sets* $\{\mathscr{Z}, \mathscr{J}\}$ *with* $\mathscr{Z} \subset \mathscr{X}$, $\mathscr{J} \subset \mathscr{S}$ *such that*[5]

$$\phi(\mathscr{Z}, \mathscr{J}) \subset \mathscr{J}. \tag{16}$$

The pair $\{\mathscr{Z}, \mathscr{J}\}$ *is invariant with respect to* Σ *if, for every* $\bar{x}$ *on the boundary of* $\mathscr{Z}$, *at least one of the following conditions holds*:

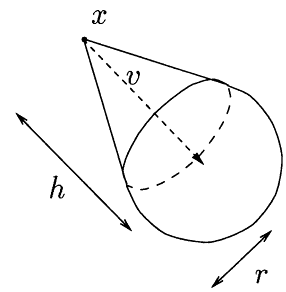

Fig. 4. Cone with spherical base $\mathscr{C}[\mathbf{v}, h, r]$.

(1) $\bar{x} \in \mathscr{Z}$ *and* $f_j(\bar{x}) = 0$ *for every* $j \in \phi(\{\bar{x}\}, \mathscr{J})$.
(2) $\bar{x} \in \mathscr{Z}$ *and* $\mathbf{v}_j \triangleq (\bar{x}, f_j(\bar{x}))$ *points towards* $\mathscr{Z}$ *for each* $j \in \phi(\{\bar{x}\}, \mathscr{J})$.
(3) $\bar{x} \notin \mathscr{Z}$ *and there exists a neighborhood* $\mathscr{N}_{\bar{x}}$ *of* $\bar{x}$ *such that* $\mathbf{v}_j \triangleq (\bar{x}, -f_j(\bar{x}))$ *points towards* $\mathscr{X} \backslash \mathscr{Z}$ *for each* $j \in \phi(\mathscr{N}_{\bar{x}} \cap \mathscr{Z}, \mathscr{J})$.

Proof of Lemma 3. By contradiction assume that there exists a solution $\{x, \sigma\}$ to Σ on $[t_0, T)$ $(T \leq +\infty)$, with $x(t_0) = x_0 \in \mathscr{Z}$ and $\sigma^-(t_0) = \sigma_0 \in \mathscr{J}$, such that

$$\bar{t} \triangleq \inf\{t \in [t_0, T) : x(t) \notin \mathscr{Z} \text{ or } \sigma(t) \notin \mathscr{J}\} \tag{17}$$

is strictly smaller than T. The vector $\bar{x} \triangleq x(\bar{t})$ cannot be in the interior of the complement of $\mathscr{Z}$, otherwise, by continuity of x, there would be a time $t < \bar{t}$ for which $x(t) \notin \mathscr{Z}$, which contradicts Eq. (17). Suppose now that $\bar{x}$ is in the interior of $\mathscr{Z}$ and therefore, by continuity, that

$$x(t) \in \mathscr{Z}, \quad \forall t \in [\bar{t}, \bar{t} + \delta_1) \tag{18}$$

for some $\delta_1 > 0$. Since $\sigma^-(\bar{t})$ is still in $\mathscr{J}$,

$$\sigma(\bar{t}) = \phi(\bar{x}, \sigma^-(\bar{t})) \in \phi(\{\bar{x}\}, \mathscr{J})$$

and therefore, because of its right-continuity, σ must remain in $\phi(\{\bar{x}\}, \mathscr{J})$ for some time after $\bar{t}$. Thus, Eqs. (16), and (18) would contradict Eq. (17) and therefore $\bar{x}$ cannot belong to the interior $\mathscr{Z}$. Since $\bar{x}$ is not in the interior of $\mathscr{Z}$ nor in the interior of its complement, it must be on the boundary of $\mathscr{Z}$. We consider three cases separately:

Case 1: $\bar{x} \in \mathscr{Z}$ and $f_j(\bar{x}) = 0$ for every $j \in \phi(\{\bar{x}\}, \mathscr{J})$. Since $\sigma^-(\bar{t})$ is still in $\mathscr{J}$,

$$\sigma(\bar{t}) = \phi(\bar{x}, \sigma^-(\bar{t})) \in \phi(\{\bar{x}\}, \mathscr{J}) \tag{19}$$

and therefore, because of its right-continuity, σ must remain in $\phi(\{\bar{x}\}, \mathscr{J})$ on some interval $[\bar{t}, \bar{t} + \delta]$ of positive length. But then Eq. (14) has a unique solution $x(t) = \bar{x}$ for $t \in [\bar{t}, \bar{t} + \delta]$. Thus $x \in \mathscr{Z}$ and $\sigma \in \mathscr{J}$ on $[\bar{t}, \bar{t} + \delta]$, which contradicts Eq. (17).

Case 2: $\bar{x} \in \mathscr{Z}$ but $f_j(\bar{x}) \neq 0$ for some $j \in \phi(\{\bar{x}\}, \mathscr{J})$. Because $\bar{x} \in \mathscr{Z}$, reasoning as in Case 1 one concludes that there must then be an interval $[\bar{t}, \bar{t} + \delta]$ of positive length in which σ remains constant and equal to some $\bar{\sigma} \in \phi(\{\bar{x}\}, \mathscr{J})$. Since $f_{\bar{\sigma}}$ is continuous and $\sigma = \sigma$ on

[4] Here, a vector at a point $x \in \mathscr{X} \triangleq \mathbb{R}^n$ is a pair $\mathbf{v} = (x, v)$ where $v \in \mathscr{X}$. Geometrically, $\mathbf{v}$ can be regarded as the vector v translated so that its "tail" is at x rather than at the origin.

[5] With $\mathscr{A}$ and $\mathscr{B}$ sets, $\mathscr{C} \subset \mathscr{A}$, and $f : \mathscr{A} \to \mathscr{B}$, $f(\mathscr{C})$ denotes the f-image of $\mathscr{C}$ that is defined by $\{f(a) : a \in \mathscr{C}\}$.

$[\bar{t}, \bar{t}+\delta]$, x must be continuously differentiable on the same interval. Therefore, by the Mean Value Theorem [cf., Lang, 1989, Corollary 4.4, p. 379),

$$\| x(\bar{t}+\varepsilon) - x(\bar{t}) - \varepsilon \dot{x}(\bar{t}) \| \le \varepsilon \sup_{\tau \in [\bar{t}, \bar{t}+\varepsilon]} \| \dot{x}(\tau) - \dot{x}(\bar{t}) \|, \quad \forall \varepsilon \in [0, \delta]$$

which means that

$$\| x(\bar{t}+\varepsilon) - \bar{x} - \varepsilon f_{\bar{\sigma}}(\bar{x}) \| \le \varepsilon \sup_{\tau \in [\bar{t}, \bar{t}+\varepsilon]} \| f_{\bar{\sigma}}(x(\tau)) - f_{\bar{\sigma}}(x(\bar{t})) \|, \quad \forall \varepsilon \in [0, \delta]. \tag{20}$$

Since $f_{\bar{\sigma}}$ is locally Lipschitz and x is continuously differentiable, the composition of these two functions is locally Lipschitz. Therefore there must exist a constant c such that

$$\| f_{\bar{\sigma}}(x(\tau)) - f_{\bar{\sigma}}(\bar{x}) \| \le c \| \tau - \bar{t} \|, \quad \tau \in [\bar{t}, \bar{t}+\delta].$$

From this and Eq. (20) one concludes that

$$\| x(\bar{t}+\varepsilon) - \bar{x} - \varepsilon f_{\bar{\sigma}}(\bar{x}) \| \le c\varepsilon^2, \quad \forall \varepsilon \in [0, \delta]. \tag{21}$$

Now, since by hypothesis $\mathbf{v}_{\bar{\sigma}} \triangleq (\bar{x}, f_{\bar{\sigma}}(\bar{x}))$ points towards $\mathscr{Z}$, there must exist positive constants h and r such that $C[\mathbf{v}_{\bar{\sigma}}, h, r] \subset \mathscr{Z}$. Rewriting Eq. (21) as

$$\| x(\bar{t}+\varepsilon) - \bar{x} - \rho h f_{\bar{\sigma}}(\bar{x}) \| \le \rho c h \varepsilon, \quad \forall \varepsilon \in [0, \delta],$$

where $\rho \triangleq \varepsilon / h$, one then concludes that

$$x(\bar{t}+\varepsilon) \in C[\mathbf{v}_{\bar{\sigma}}, h, r] \subset \mathscr{Z}, \quad 0 \le \varepsilon \le \min\left\{\delta, h, \frac{r}{ch}\right\}$$

which contradicts Eq. (17) since $\sigma(\bar{t}+\varepsilon) \in \mathscr{J}$ for $\varepsilon \in [0, \delta]$.

Case 3: $\bar{x} \notin \mathscr{Z}$. By the hypothesis of the lemma, there must then exist a neighborhood $\mathscr{N}_{\bar{x}}$ of $\bar{x}$ such that $\mathbf{v}_j \triangleq (\bar{x}, -f_j(\bar{x}))$ points towards $\mathscr{X} \backslash \mathscr{Z}$ for each $j \in \phi(\mathscr{N}_{\bar{x}} \cap \mathscr{Z}, \mathscr{J})$. Since $x(t_0) \in \mathscr{Z}$ and $\bar{x} = x(\bar{t}) \notin \mathscr{Z}$, one must have $\bar{t} > t_0$ and therefore there must be an interval $[\bar{t}-\delta, \bar{t}) \subset [t_0, T)$ of positive length on which x remains in $\mathscr{Z}$. Because of the continuity of x and the right-continuity of σ, one can pick δ small enough so that x remains inside $\mathscr{N}_{\bar{x}}$ on $[\bar{t}-\delta, \bar{t})$ and σ is equal to some constant $\bar{\sigma} \in \phi(\mathscr{N}_{\bar{x}} \cap \mathscr{Z}, \mathscr{J})$ on $[\bar{t}-\delta, \bar{t})$. Since $f_{\bar{\sigma}}$ is continuous and $\sigma = \bar{\sigma}$ on $[\bar{t}-\delta, \bar{t})$, x must be continuously differentiable on the same interval. Therefore, by the Mean Value Theorem,

$$\| x(\bar{t}-\varepsilon) - x(\bar{t}) + \varepsilon \dot{x}(\bar{t}) \| \le \varepsilon \sup_{\tau \in [\bar{t}-\varepsilon, \bar{t})} \| \dot{x}(\tau) - \dot{x}(\bar{t}) \|, \quad \forall \varepsilon \in [0, \delta].$$

Proceeding as in Case 2 one can then conclude that there exists a constant c such that

$$\| x(\bar{t}-\varepsilon) - \bar{x} + \varepsilon f_{\bar{\sigma}}(\bar{x}) \| \le c\varepsilon^2, \quad \forall \varepsilon \in [0, \delta]. \tag{22}$$

Now, since $\mathbf{v}_{\bar{\sigma}} \triangleq (\bar{x}, -f_{\bar{\sigma}}(\bar{x}))$ points towards $\mathscr{X} \backslash \mathscr{Z}$, there must exist positive constants h and r such that $C[\mathbf{v}_{\bar{\sigma}}, h, r] \subset \mathscr{X} \backslash \mathscr{Z}$. Rewriting Eq. (22) as

$$\| x(\bar{t}-\varepsilon) - \bar{x} + \rho h f_{\bar{\sigma}}(\bar{x}) \| \le \rho c h \varepsilon, \quad \forall \varepsilon \in [0, \delta],$$

where $\rho \triangleq \varepsilon / h$, one then concludes that

$$x(\bar{t}-\varepsilon) \in C[\mathbf{v}_{\bar{\sigma}}, h, r] \subset \mathscr{X} \backslash \mathscr{Z}, \quad 0 \le \varepsilon \le \min\left\{\delta, h, \frac{r}{ch}\right\},$$

which contradicts Eq. (17). □

5. Proof of Theorem 2

Consider the sets $\mathscr{J} \triangleq \{2, 3, 4\}$,

$$\mathscr{Z}_1 \triangleq \{x \in \mathbb{R}^3 : x_3^2 \le c_1, \pi_1(x_3^2) < x_1^2 + x_2^2 \le c_2\} \cup \{0\}, \tag{23}$$

$$\mathscr{Z}_2 \triangleq \{x \in \mathbb{R}^3 : \pi_3(x_3^2) \le x_1^2 + x_2^2\}, \tag{24}$$

$$\mathscr{Z}_3 \triangleq \{x \in \mathbb{R}^3 : x_3^2 \le c, \pi_3(x_3^2) \le x_1^2 + x_2^2 \le \pi_4(x_3^2)\}, \tag{25}$$

with c_1 an arbitrary constant, c_2 a constant larger than $\pi_4(c_1)$, and c as in Eq. (3). Setting $w \triangleq \Pi(x)$, with $\Pi : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $[x_1\ x_2\ x_3]' \mapsto [x_3^2\ x_1^2 + x_2^2]'$, when σ takes values on $\mathscr{J}$, the evolution of w is completely determined by the hybrid system Σ_w defined[6] by

$$\dot{w} = f_\sigma(w), \qquad \sigma = \varphi(w, \sigma^-), \tag{26}$$

where, for every $w \in \mathbb{R}^2$ and every $j \in \{2, 3, 4\}$,

$$f_j(w) \triangleq \begin{cases} [-2|w_1|\ \ +2w_2]', & w_2 \ge 0, \quad j = 2, \\ [-2|w_1|\ \ -2w_2]', & w_2 \ge 0, \quad j \in \{3, 4\}, \\ [-2|w_1|\ \ 0]', & w_2 < 0, \end{cases}$$

$$\varphi(w, j) \triangleq \begin{cases} j, \quad w \in \Pi(R_j) \text{ or } w_1 < 0 \text{ or } w_2 < 0, \\ \max\{i \in \mathscr{S} : w \in \Pi(R_i)\}, \quad w \notin \Pi(\mathscr{R}_j) \text{ and } w_1 \ge 0 \text{ and } w_2 \ge 0. \end{cases}$$

Moreover, defining

$$\mathscr{W}_1 \triangleq \{w : w_1 \in (0, c_1], \pi_1(w_1) < w_2 \le c_2\} \cup \{w \in \mathbb{R}^2 : w_1 \le 0, w_2 \le c_2\},$$

$$\mathscr{W}_2 \triangleq \{w : w_1 \ge 0, w_2 \ge \pi_3(w_1)\} \cup \{w \in \mathbb{R}^2 : w_1 \le 0\},$$

$$\mathscr{W}_3 \triangleq \{w : w_1 \in [0, c], \pi_3(w_1) \le w_2 \le \pi_4(w_1)\},$$

a vector $x \in \mathbb{R}^3$ belongs to $\mathscr{Z}_i$ just in case $w \triangleq \Pi(x)$ belongs to $\mathscr{W}_i$. Therefore, each pair $\{\mathscr{Z}_i, J\}$ is invariant with respect to the hybrid system Σ defined by Eqs. (2), (4), and (7), if the pair $\{\mathscr{W}_i, \mathscr{J}\}$ is invariant with respect to

[6] The values of the vector fields f_j and the discrete transition function φ, when either $w_1 < 0$ or $w_2 < 0$, are arbitrary. The definitions given simplify somewhat the analysis of Σ_w.

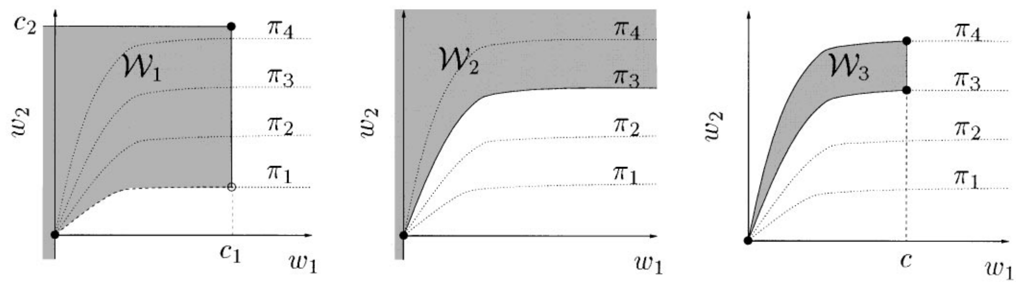

Fig. 5. Sets $\mathscr{W}_1$, $\mathscr{W}_2$, and $\mathscr{W}_3$.

the hybrid system Σ_w defined by Eq. (26). A straightforward application of Lemma 3 allows one to conclude that each pair $\{\mathscr{W}_i, \mathscr{J}\}$ is indeed invariant with respect to Σ_w and one thus concludes the following:

Lemma 4. *Each of the pairs* $\{\mathscr{X}_i, \mathscr{J}\}$ *with* $i \in \{1, 2, 3\}$, *is invariant with respect to the hybrid system* Σ *defined by Eqs.* (2), (4) *and* (7).

It was argued in Section 3 that for every initialization, the system Σ defined by Eqs. (2), (4) and (7) has a unique solution that exists globally. In the sequel let $\{x, \sigma\}$ denote such a solution defined on the interval $[t_0, \infty)$.

Lyapunov stability. To prove that the origin is a Lyapunov stable equilibrium point of Σ it is enough to show that by making $\|x(t_0)\|$ small enough it is possible to guarantee that $x(t)$ remains in a ball around the origin of arbitrarily small radius for all $t \geq t_0$. We consider two cases separately:

$\sigma(t_0) \in \mathscr{J} \triangleq \{2, 3, 4\}$: Because of Eqs. (7) and (8), $x(t_0)$ must belong to $\mathscr{R}_{\sigma(t_0)}$, which, since $\sigma(t_0)$ is equal to 2, 3, or 4, implies that either $\pi_1(x_3(t_0)^2) < x_1(t_0)^2 + x_2(t_0)^2$ or $x(t_0) = 0$. Therefore $x(t_0)$ belongs to the set $\mathscr{X}_1$ defined by (23) with $c_1 \triangleq \|x(t_0)\|^2$ and $c_2 \triangleq \|x(t_0)\|^2 + \pi_4(\|x(t_0)\|^2)$. Since the pair $\{\mathscr{X}_1, \mathscr{J}\}$ is invariant with respect to Σ one concludes that $x(t) \in \mathscr{X}_1$ for every $t \geq t_0$. From this and Eq. (23) one concludes that

$$\|x(t)\|^2 \leq c_1 + c_2 = 2\|x(t_0)\|^2 + \pi_4(\|x(t_0)\|^2), \quad \forall t \geq t_0. \tag{27}$$

$\sigma(t_0) = 1$: While $\sigma = 1$, $u = g_1(x)$ and therefore

$$x_i(t) = x_i(t_0) + t - t_0, \quad i \in \{1, 2\}, \tag{28}$$

$$x_3(t) = x_3(t_0) + (x_1(t_0) - x_2(t_0))(t - t_0). \tag{29}$$

In case $x_1(t_0) = x_2(t_0) < 0$ and $x_3(t_0) = 0$ then x becomes zero in finite time and

$$\|x(t)\| \leq \|x(t_0)\|, \quad \forall t \geq t_0. \tag{30}$$

Otherwise, $x_1^2 + x_2^2$ grows quadratically with t and, since in $\mathscr{R}_1$ the signal $x_1^2 + x_2^2$ must be bounded, one concludes that x leaves $\mathscr{R}_1$ at some finite time t_1. At this time, σ will switch from 1 to 2. By intersecting the trajectory given by Eqs. (28) and (29) with the boundary of $\mathscr{R}_1$ it is straightforward to conclude that

$$\|x(t)\|^2 \leq k\|x(t_0)\|^2 + \pi_2(k\|x(t_0)\|^2), \quad t \in [t_0, t_1], \tag{31}$$

with $k \triangleq (2 + 8\sup_{w \geq 0}\pi_2(w))$, and that $x(t_1)$ belongs to the set $\mathscr{X}_1$ defined by Eq. (23) with $c_1 = k\|x(t_0)\|^2$ and $c_2 = 2\pi_4(k\|x(t_0)\|^2)$. Since the pair $\{\mathscr{X}_1, \mathscr{J}\}$ is invariant with respect to Σ, one concludes that $x(t) \in \mathscr{X}_1$ for every $t \geq t_1$. From this, Eqs. (23) and (31) one concludes that

$$\|x(t)\|^2 \leq k\|x(t_0)\|^2 + 2\pi_4(k\|x(t_0)\|^2), \quad t \geq t_0. \tag{32}$$

Note that this inequality holds even in the case when $x_1(t_0) = x_2(t_0) < 0$ and $x_3(t_0) = 0$ (cf. (30)).

Finally, from both Eqs. (27) and (32), it is clear that by making $\|x(t_0)\|$ small enough it is possible to guarantee that $x(t)$ remains in a ball around the origin of arbitrarily small radius for all $t \geq t_0$. This proofs that the origin is a Lyapunov stable equilibrium point of Σ. □

Exponential convergence. It was shown above that there exists a finite time t_1 after which x and σ enter the sets $\mathscr{X}_1$ and $\mathscr{J}$, respectively[7]. Since $\sigma(t) \in \mathscr{J}$ for $t \geq t_1$, defining $w \triangleq \Pi(x)$ one concludes that

$$\dot{w}_1 = -2w_1, \quad \forall t \geq t_1, \tag{33}$$

which means that $w_1 \to 0$ as fast as e^{-2t}. For $t \geq t_1$, while $x(t)$ is outside $\mathscr{X}_2$, $\sigma(t) = 2$ and therefore $\dot{w}_2 = 2w_2$. Thus, after some finite time $t_2 \geq t_1$, w_2 becomes larger or equal to $\pi_3(w_1(t_1))$. Since π_3 is a monotone nondecreasing function and, because of Eq. (33), w_1 is also monotone nondecreasing,

$$w_2(t_2) \geq \pi_3(w_1(t_1)) \geq \pi_3(w_1(t_2)).$$

Therefore $x(t_2)$ is inside the set $\mathscr{X}_2$ defined by Eq. (24). Since $\sigma(t_2) \in \mathscr{J}$ and the pair $\{\mathscr{X}_2, \mathscr{J}\}$ is invariant with respect to Σ, $x(t)$ remains in $\mathscr{X}_2$ for $t \geq t_2$. Since it has been established that w_1 converges to zero, without loss

[7] When $\sigma(t_0) \in \mathscr{J}$, one can just take $t_1 = t_0$.

of generality one can assume that t_2 is large enough so that $w_1(t_2) \le c$. After t_2 two cases are possible:

Case 1: x gets into the set $\mathscr{X}_3$ at some finite time t_3. Since $\sigma(t_3) \in \mathscr{J}$ and the pair $\{\mathscr{X}_3, \mathscr{J}\}$ is invariant with respect to Σ, $x(t)$ remains in $\mathscr{X}_3$ for $t \ge t_2$ and therefore

$$w_2(t) \le \pi_4(w_1(t)), \quad \forall t \ge t_3.$$

Because of the properties of π_4, since w_1 converges to zero exponentially fast, w_2 also converges to zero exponentially fast.

Case 2: x remains in $\mathscr{X}_2 \backslash \mathscr{X}_3$. In this region, and for $w_2 \le c$, σ can only be equal to 3 and therefore

$$\dot{w}_2 = -2w_2, \quad \forall t \ge t_3.$$

Also in this case w_2 converges to zero exponentially fast.

In either case, since $\|x\| = \sqrt{w_1 + w_2}$, exponential convergence to zero of x is achieved. As for the control signal, simple algebra shows that for $t \ge t_1$

$$\|u\| \le 2\left(\sqrt{w} + \sqrt{{w_1}_{w_2}}\right) \le 2\left(\sqrt{w_2} + \sqrt{\frac{w_1}{\pi_1(w_1)}}\right). \quad (34)$$

Since w_1 and w_2 converge to zero exponentially fast, because of Eq. (34) and the properties of π_1, $\|u\|$ also converges to zero exponentially fast. □

Remark 5. It was seen above that if $\{x, \sigma\}$ gets into $\mathscr{X}_3 \times \mathscr{J}$ at some finite time t_3 then σ may switch forever between 2 and 3 (Case 1 in the proof above with $x(t_3) \ne 0$). Suppose this happens and let $\bar{t} \ge t_3$ denote an arbitrary time instant at which σ switches from 2 to 3. Then one must have

$$w_2(\bar{t}) = \pi_4(w_1(\bar{t})) \quad (35)$$

and σ can only switch back to 2 after some time interval Δt for which

$$w_2(\bar{t} + \Delta t) = \pi_3(w_1(\bar{t} + \Delta t)). \quad (36)$$

Since $\dot{w}_2 = -2w_2$ on the interval $[\bar{t}, \bar{t} + \Delta t)$,

$$w_2(\bar{t} + \Delta t) = w_2(\bar{t})\mathrm{e}^{-2\Delta t}.$$

From this, Eqs. (35) and (36), one concludes that

$$\Delta t = \tfrac{1}{2} \log \frac{\pi_4(w_1(\bar{t}))}{\pi_3(w_1(\bar{t} + \Delta t))}. \quad (37)$$

But w_1 is decreasing for all $t \ge t_3$ and π_3 is monotone nondecreasing, thus $\pi_3(w_1(\bar{t} + \Delta t)) \le \pi_3(w_1(\bar{t}))$. From this and Eq. (37) one concludes that

$$\Delta t \ge \tfrac{1}{2} \log \frac{\pi_4(w_1(\bar{t}))}{\pi_3(w_1(\bar{t}))}.$$

Thus, for the π_j defined by Eqs. (10) and (11), any time interval for which σ remains constant equal to 3 is bounded below by $\frac{1}{2}\log\frac{4}{2.5}$. A lower bound on any time interval for which σ remains constant equal to 2 can be computed in a similar fashion. One thus concludes that, not only is chattering precluded on any finite time interval, but also that the interval between consecutive switchings is bounded away from zero as time goes to infinity.

6. Conclusion

In this paper it is shown that time-invariant logic-based switching can be used to effectively control nonholonomic systems. Arguments based on set invariance were used to prove Lyapunov stability and exponential convergence of the state of the nonholonomic integrator to the origin. Simulation experiments show that simple "actuator dynamics" do not compromise the exponential convergence nor do they introduce chattering.

The performance of the closed-loop system, in terms of speed of convergence and magnitude of the control signals, seems to be at least as good as the one obtained with time-varying controllers that achieve exponential convergence (e.g. M'Closkey and Murray, 1997). We believe that definite advantages/drawbacks of time-varying controllers over hybrid control laws can only be investigated in concrete applications.

The control law proposed can be generalized to higher-dimensional nonholonomic integrators like the ones considered by Hespanha (1996). Further effort is being made to design similar control laws for other types of nonholonomic systems. The use of hybrid control laws also seems promising in the control of nonholonomic systems with parametric uncertainty (Hespanha et al., 1998). Another question that deserves attention is prompted by Teel's (1996) observation that for hybrid systems like the one proposed in this paper, the classical solution to the continuous dynamics varies discontinuously with respect to continuous variations of the initial state, therefore leading to the *hidden possibility of indecision.*

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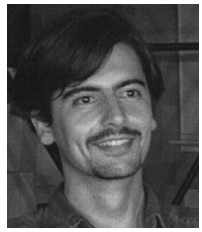

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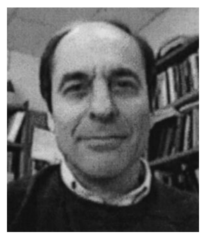

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