A. S. Morse

Center for Computational Vision and Control
Department of Electrical Engineering, Yale University
New Haven, Connecticut, 06520-208267

A Bound for the Disturbance - to - Tracking - Error Gain of a Supervised Set-Point Control System *

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Dedicated to Ioan D. Landau on the Occasion of His Sixtieth Birthday

Summary.

The aim of this paper is to provide a simple analysis of the dynamical behavior of a set-point control system consisting of a poorly modelled process, an integrator and a multi-controller supervised by an estimator-based algorithm employing dwell-time switching. For a slowly switched multi-controller implementation of a finite family of linear controllers, explicit upper bounds are derived for the normed-value of the process's allowable unmodelled dynamics as well as for the system's disturbance-to-tracking error gain.

0.1 Introduction

Much has happened in adaptive control since Ioan Landau published his pioneering monograph in 1979 [1]. The solution to the classical model reference problem is by now well understood. Provably correct algorithms exist which, at least in theory, are capable of dealing with unmodelled dynamics, noise, right-half-plane zeros, and even certain types of nonlinearities – and a number of excellent text and monographs have been written covering many of these advances [2, 3, 4, 5, 6, 7, 8].

However despite the impressive gains made since 1979, there remain many important, unanswered questions: Why, for example, is it still so difficult to explain to a novice why a particular algorithm is able to functions correctly in the face of unmodelled process dynamics and $\mathcal{L}^\infty$ bounded noise? How much unmodelled dynamics can a given algorithm tolerate before loop-stability is lost? How do we choose an adaptive control algorithm's many design parameters to achieve good disturbance rejection, transient response, etc.?

It is our view that eventually there will be satisfactory answers to all of these questions, that adaptive control will become much more accessible to non-specialists, that we will be able to much more clearly and concisely quantify unmodelled dynamics norm bounds, disturbance-to-controlled output gains, and so on and that because of this we will see the emergence of a bona fide computer-aided adaptive control design methodology which relies much more on design principals then on trial and error techniques. It is with these ends in mind, that this paper has been written.

In the sequel we provide a relatively uncluttered analysis of the dynamical behavior of a set-point control system consisting of a poorly modelled process, an integrator and a multi-controller supervised by an estimator-based algorithm employing dwell-time switching. The system has been considered previously in [9]. It has been analyzed in one form or another in [10, 11, 12, 13] and elsewhere under various assumptions. It has been shown in [12] that the system's supervisor can successfully orchestrate the switching of a sequence of candidate set-point controllers into feedback with the system's imprecisely modeled siso process so as (i) to cause the output of the process to approach and track a constant reference input despite norm-bounded unmodelled dynamics, and constant process disturbances and (ii) to insure that none of the signals within the overall system can grow without bound in response to bounded disturbance, be they constant or not. The objective of this paper is to re-derive these same results in a much more straight forward manner. This will be done for a supervisory control system in which the number of candidate controllers is finite, and the switching between candidate controllers is constrained to be "slow" in a sense to be made precise in the sequel. These restrictions not only greatly simplify the analysis in comparison with that given in [12], but also make it possible to derive reasonably explicit upper bounds for the process's allowable unmodelled dynamics as well as for the system's disturbance-to-tracking error gain.

The overall supervisory control system to be considered is described in §2. The main theorem characterizing the system's behavior is re-stated in §3. A simple, informal proof of the theorem is carried out in §4. Explicit bounds for the process's allowable unmodelled dynamics as well as for the system's disturbance-to-tracking error gain appear (0.22) and (0.24) respectively.

0.2 The Overall System

The aim of this section is to describe the structure of the supervisory control system to be considered in this paper. We begin with a description of the process.

0.2.1 The Process

The overall problem of interest is to construct a control system capable of driving to and holding at a prescribed set-point r, the output of a process modeled by a dynamical system with large scale uncertainty. The process is presumed to admit the model of a siso linear system Σ_P whose transfer function from control input u to measured output y is a member of a known class of admissible transfer functions of the form

$$\mathcal{C}_P = \bigcup_{p \in \mathcal{P}} \{\nu_p + \delta : \| \delta \| \leq \epsilon_p\}$$

where $\mathcal{P}$ is a finite set of indices,

$$\nu_p \stackrel{\Delta}{=} \frac{\alpha_p}{\beta_p}$$

is a prespecified, strictly proper, *nominal transfer function*, ϵ_p is a real nonnegative number, δ is a proper stable transfer function whose poles all have real parts less than the negative of a prespecified *stability margin* $\lambda > 0$, and $\| \cdot \|$ is the shifted infinity norm

$$\|\delta\| = \sup_{\omega \in \mathbb{R}} |\delta(j\omega - \lambda)|$$

It is assumed for each $p \in \mathcal{P}$, that β_p is monic and that α_p and β_p are coprime. All transfer functions in $\mathcal{C}_P$ are thus proper, but not necessarily stable rational functions. Prompted by the requirements of set-point control, it is further assumed that the numerator of each transfer function in $\mathcal{C}_P$ is nonzero at $s = 0$. The specific model of the process to be controlled is shown in Figure 0.1.

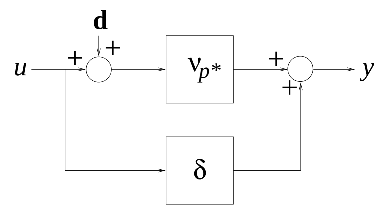

Fig. 0.1. Process Model

Here y is the process's measured output and $\mathbf{d}$ is a disturbance.

0.2.2 The System to Be Supervised

Presumed given is an indexed family of "off-the-shelf" loop controller transfer functions $\mathcal{K} \stackrel{\Delta}{=} \{\kappa_p : p \in \mathcal{P}\}$ with at least the following property:

Stability Margin Property: *For each $p \in \mathcal{P}$, $-\lambda$ is greater than the real parts of all of the closed-loop poles*[1] *of the feedback interconnection*

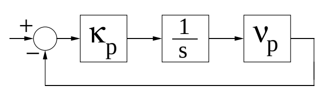

Fig. 0.2. Feedback Interconnection

Also presumed given is an integer $n_C \geq 0$ and a family of n_C-dimensional realizations $\{A_p, b_p, f_p, g_p\}$, one for each $\kappa_p \in \mathcal{K}$. These realizations are required to be chosen so that for each $p \in \mathcal{P}$, $(c_p, \lambda I + A_p)$ is detectable and $(\lambda I + A_p, b_p)$ is stabilizable. As noted in [9], there are a great many different ways to construct such realizations, once one has in hand an upper bound n_κ on the McMillan Degrees of the κ_p. Given such a family of realizations, the sub-system to be supervised is thus of the form shown in Figure 0.3 where

[1] By the closed-loop poles are meant the zeros of the polynomial $s\rho_p\beta_p + \gamma_p\alpha_p$, where $\frac{\alpha_p}{\beta_p}$ and $\frac{\gamma_p}{\rho_p}$ are the reduced transfer functions ν_p and κ_p respectively.

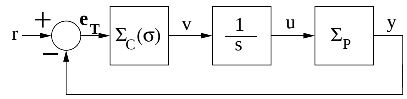

Fig. 0.3. Supervised Sub-System

$\Sigma_C(\sigma)$ is the n_C-dimensional "state-shared" dynamical system

$$\dot{x}_C = A_\sigma x_C + b_\sigma \mathbf{e_T} \qquad v = f_\sigma x_C + g_\sigma \mathbf{e_T}, \tag{0.1}$$

called a *multi-controller*, v is the input to the *integrator*

$$\dot{u} = v, \tag{0.2}$$

$\mathbf{e_T}$ is the *tracking error*

$$\mathbf{e_T} \stackrel{\Delta}{=} r - y, \tag{0.3}$$

and σ is a piecewise constant *switching signal* taking values in $\mathcal{P}$.

0.2.3 The Supervisor

The problem of interest is to construct a provably correct "supervisor" which is capable of generating σ so as to achieve 1. *global boundedness* {of all system signals} in the face of an arbitrary but bounded disturbance input and 2. *set-point regulation* {i.e., $\mathbf{e_T} \to 0$} in the event that the disturbance signal is constant. The functioning of the supervisor to be considered can be explained informally in terms of the "multi-estimator" architecture shown in Figure 0.4. Here each y_p is a suitably defined

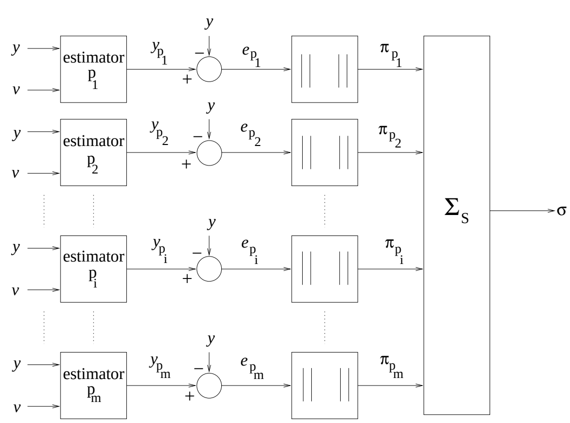

Fig. 0.4. Estimator-Based Supervisor

estimate of y which would be asymptotically correct if ν_p were the process model's transfer function and there were no noise or disturbances. For each $p \in \mathcal{P}$, $e_p = y_p - y$ denotes the pth output estimation error and π_p is a "normed" value of e_p or a "performance signal" which is used by the supervisor assess the potential performance of controller p. Σ_S is a switching logic whose function is to determine σ

on the basis of the current values of the π_p. The underlying decision making strategy used by such a supervisor is basically this: From time to time select for σ, that candidate control index q whose corresponding performance signal π_q is the smallest among the $\pi_p,\ p \in \mathcal{P}$. Motivation for this idea is obvious: the nominal process model whose associated performance signal is the smallest, "best" approximates what the process is and thus the candidate controller designed on the basis of that model ought to be able to do the best job of controlling the process.

The specific supervisor considered below admits a slightly different realization than that shown in Figure 0.4, even though the two supervisors are input-output equivalent. Internally the supervisor we want to discuss consists of three subsystems: a *multi-estimator dynamic* Σ_E, a *performance-weight generator* Σ_W, and a *dwell-time switching logic* Σ_D.

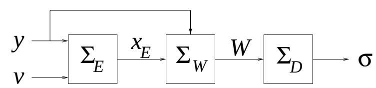

Fig. 0.5. Estimator-Based Supervisor

Σ_E is a n_E-dimensional linear dynamical system of the form

$$\dot{x}_E = \begin{bmatrix} A_E & 0 \\ 0 & A_E \end{bmatrix} x_E + \begin{bmatrix} b_E \\ 0 \end{bmatrix} y + \begin{bmatrix} 0 \\ b_E \end{bmatrix} v \tag{0.4}$$

where $n_E \triangleq 2(n_\nu + 1)$ and (A_E, b_E) is a parameter-independent, $n_\nu + 1$-dimensional siso, controllable pair with $\lambda I + A_E$ stable. Here n_ν is an upper bound on the McMillan Degrees of the $\nu_p,\ p \in \mathcal{P}$. In [9] it is explained how to construct a function $p \longmapsto c_p$ so that for each $p \in \mathcal{P}$,

$$\left\{ \begin{bmatrix} A_E & 0 \\ 0 & A_E \end{bmatrix} + \begin{bmatrix} b_E \\ 0 \end{bmatrix} c_p,\ \begin{bmatrix} 0 \\ b_E \end{bmatrix},\ c_p \right\}$$

is a stabilizable realization of $\frac{1}{s}\nu_p$ whose uncontrollable eigenvalues have real parts less than $-\lambda$. The c_p are used in the definition of Σ_W which will be given in a moment. The c_p also enable us to define *output estimation errors*

$$e_p \triangleq c_p x_E - y, \quad p \in \mathcal{P} \tag{0.5}$$

While these error signals are not actually generated by the supervisor, they play an important role in explaining how the supervisor functions.

The supervisor's second subsystem, Σ_W, is a causal dynamical system whose inputs are x_E and y and whose state and output W is a "weighting matrix" which takes values in a linear space $\mathcal{W}$. W together with a suitably defined *performance function* $\Pi : \mathcal{W} \times \mathcal{P} \rightarrow \mathbb{R}$ determine a scalar-valued *performance signal* of the form

$$\pi_p \triangleq \Pi(W, p) \tag{0.6}$$

which is viewed by the supervisor as a measure of the expected performance of controller p. Σ_W and Π are defined by

$$\dot{W} = -2\lambda W + \begin{bmatrix} x_E \\ y \end{bmatrix} \begin{bmatrix} x_E \\ y \end{bmatrix}' \tag{0.7}$$

and

$$\Pi(W,p) = [\,c_p \quad -1\,]\,W\,[\,c_p \quad -1\,]' \tag{0.8}$$

respectively. The definitions of Σ_W and Π are prompted by the observation that if π_p are given by (0.6), then

$$\dot{\pi}_p = -2\lambda\pi_p + e_p^2, \quad p \in \mathcal{P} \tag{0.9}$$

because of (0.5), (0.7) and (0.8).

The supervisor's third subsystem, called a *dwell-time switching logic* Σ_D, is a hybrid dynamical system whose input and output are W and σ respectively, and whose state is the ordered triple $\{X, \tau, \sigma\}$. Here X is a discrete-time matrix which takes on sampled values of W, and τ is a continuous-time variable called a *timing signal*. τ takes values in the closed interval $[0, \tau_D]$, where τ_D is a prespecified positive number called a *dwell time*. Also assumed prespecified is a *computation time* $\tau_C \leq \tau_D$ which bounds from above for any $X \in \mathcal{W}$, the time it would take a supervisor to compute a value $p = p_X \in \mathcal{P}$ which minimizes $\Pi(X,p)$. Between "event times," τ is generated by a reset integrator according to the rule $\dot{\tau} = 1$. Event times occur when the value of τ reaches either $\tau_D - \tau_C$ or τ_D; at such times τ is reset to either 0 or $\tau_D - \tau_C$ depending on the value of Σ_D's state. Σ_D's internal logic is defined by the computer diagram shown in Figure 0.6 where p_X denotes a value of $p \in \mathcal{P}$ which minimizes $\Pi(X,p)$.

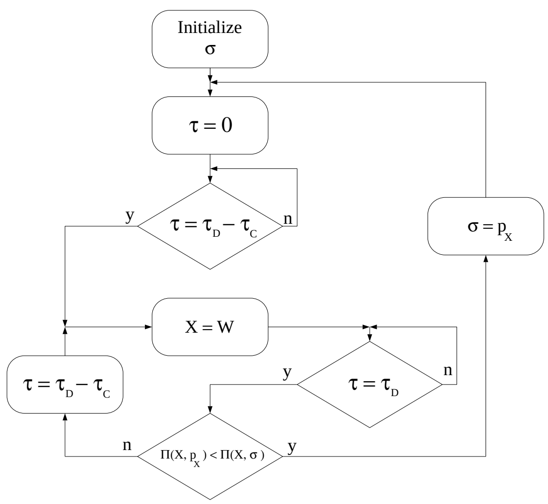

Fig. 0.6. Computer Diagram of Σ_D

In the sequel we call a piecewise-constant signal $\bar{\sigma} : [0, \infty) \to \mathcal{P}$ *admissible* if it either switches values at most once, or if it switches more than once and the set of time differences between each two successive switching times is bounded below by τ_D. We write $\mathbb{S}$ for the set of all admissible switching signals. Because of the definition of Σ_D, it is clear its output σ will be admissible. This means that switching cannot occur infinitely fast and thus that existence and uniqueness of solutions to the differential equations involved is not an issue.

0.3 Discussion

The overall system just described, admits a block diagram description of the form

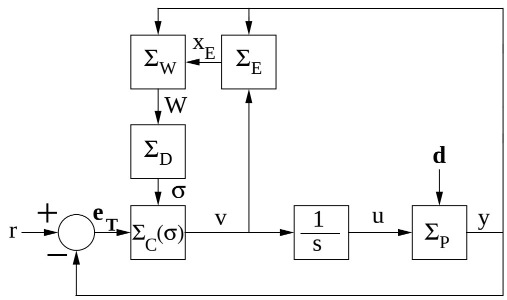

Fig. 0.7. Supervisory Control System

The following theorem is proved in [12]:

Theorem 0.3.1. *Let $\tau_C \geq 0$ be fixed. Let τ_D be any positive number no smaller than τ_C. There are positive numbers ϵ_p, $p \in \mathcal{P}$, for which the following statements are true provided Σ_P has a transfer function in $\mathcal{C}_P$.*

1. **Global Boundedness:** *For each constant set-point value r, each bounded piecewise-continuous disturbance input $\mathbf{d}$, and each system initialization, u, x_C, x_E, W, and X are bounded responses.*
2. **Tracking and Disturbance Rejection:** *For each constant set-point value r, each constant disturbance $\mathbf{d}$, and each system initialization, y tends to r and u, x_C, x_E, W, and X tend to finite limits, all as fast as $e^{-\lambda t}$.*

The theorem implies that the overall supervisory control system shown in Figure 0.7 has the basic properties one would expect of a non-adaptive set-point control system.

0.4 Analysis

The aim of this section is to re-derive Theorem 0.3.1 in a much more straight forward manner than in [12]. This will be done for a supervisory control system in which the switching between candidate controllers is constrained to be "slow" in a sense to be made precise in §0.4.1. This restrictions not only greatly simplifies the analysis, but also make it possible to derive reasonably explicit bounds for the process's allowable unmodelled dynamics as well as for the system's disturbance-to-tracking-error gain.

In the sequel we will invariably ignore initial condition dependent terms which decay to zero as fast as $e^{-\lambda t}$, as this will make things much easier to follow. A more thorough analysis which would take these terms into account can carried out in essentially the same manner.

0.4.1 Slow Switching

Assume that r is a constant and let x denote the composite state

$$x = \begin{bmatrix} \bar{x}_E \\ x_C \end{bmatrix} \tag{0.10}$$

where $\bar{x}_E$ is the shifted state

$$\bar{x}_E = x_E + \begin{bmatrix} A_E^{-1} b_E \\ 0 \end{bmatrix} r \tag{0.11}$$

It is then possible to show in a straightforward manner, that for any $q \in \mathcal{P}$ and any given piecewise constant switching signal $\sigma : [0, \infty) \to \mathcal{P}$, whether generated by Σ_D or not, the relationships between e_q, e_σ, v, and $\mathbf{e_T}$ determined by (0.1)-(0.5) are given by a system of equations of the form

$$\left.\begin{array}{rcl} e_\sigma & = & c_{\sigma q} x + e_q \\ \dot{x} & = & A_{\sigma\sigma} x + h_\sigma e_\sigma \\ v & = & f_{\sigma\sigma} x + g_\sigma e_\sigma \\ \mathbf{e_T} & = & e_\sigma - \bar{c}_\sigma x \end{array}\right\} \tag{0.12}$$

where $d \triangleq \text{column}\{-b_E,\ 0,\ 0,\ b_C\}$, $b \triangleq \text{column}\{0,\ b_E,\ b_C,\ 0\}$, and for all $p, l \in \mathcal{P}$,

$$f_{pl} \triangleq [\,-g_p c_l \quad f_p\,] \qquad h_p \triangleq b g_p + d \qquad c_{pl} \triangleq [\,c_p - c_l \quad 0\,] \qquad \bar{c}_l \triangleq [\,c_l \quad 0\,]$$

and

$$A_{pl} \triangleq \text{block diagonal}\ \{A_E,\ A_E,\ A_C,\ A_C\} - d\,[\,c_l \quad 0\,] + b f_{pl}$$

One readily verifiable and important property of the matrices defined above is that for each $p, l \in \mathcal{P}$, $(c_{pl}, \lambda I + A_{pl})$ is a detectable matrix pair [9]. This is a consequence of certainty equivalence, the Stability Margin Property, the requirements that $\lambda I + A_E$ be a stability matrix and that for $p \in \mathcal{P}$, $\{\lambda I + A_p,\ b_p,\ f_p,\ g_p\}$ be a stabilizable and detectable system. Since $c_{pp} = 0,\ p \in \mathcal{P}$, this means that for each such p, $\lambda I + A_{pp}$ must be a stability matrix [9]. In the sequel we will assume the following.

Slow Switching Assumption: *The dwell time τ_D is large enough so that for each admissible switching signal $\sigma : [0, \infty) \to \mathcal{P}$, $\lambda I + A_{\sigma\sigma}$ is an exponentially stable matrix.*

It is possible to compute an explicit lower bound for τ_D for which this assumption holds [9].

0.4.2 Norms

It is especially useful to introduce the following. For any piecewise-continuous function $z : [0, \infty) \to \mathbb{R}^n$, and any times $t_2 > t_1 \geq 0$, let us write $||z||_{\{t_1, t_2\}}$ for the exponentially weighted 2-norm

$$||z||_{\{t_1,t_2\}} \triangleq \sqrt{\int_{t_1}^{t_2} e^{2\lambda t} |z(t)|^2 dt}$$

Note that $e^{2\lambda T}\pi_p(T) = ||e_p||^2_{\{0,T\}},\ T \geq 0,\ p \in \mathcal{P}$, because of (0.9) and the assumption that $W(0) = 0$.

For any time-varying SISO linear system Σ of the form $y = c(t)x + d(t)u$, $\dot{x} = A(t)x + b(t)u$ we write

$$\left\| \begin{matrix} A & b \\ c & d \end{matrix} \right\|$$

for the induced norm $\sup\{||y_u||_{\{0,\ \infty\}} : u \in \mathcal{U}\}$ where y_u is Σ's zero initial state, output response to u and $\mathcal{U}$ is the space of all piecewise continuous signals u such that $||u||_{\{0,\ \infty\}} = 1$. The induced norm of Σ is finite whenever $\lambda I + A(t)$ is {uniformly} exponentially stable.

We note the following easily derived facts. If $e^{-\lambda t}||u||_{\{0,t\}}$ is bounded on $[0, \infty)$ {in the $\mathcal{L}^\infty$ sense}, then so is y_u provided $d = 0$ and $\lambda I - A(t)$ is exponentially stable. If u is bounded on $[0, \infty)$ in the $\mathcal{L}^\infty$ sense, then so is $e^{-\lambda t}||u||_{\{0,t\}}$. If $u \to 0$ as $t \to \infty$, then so does $e^{-\lambda t}||u||_{\{0,t\}}$.

0.4.3 Block Diagrams

Using the diagram of Σ_P in Figure 0.1 together with (0.1)-(0.5) and (0.12), it is not difficult to verify that, up to initial condition dependent terms decaying to zero as fast at $e^{-\lambda t}$, the relationships between $\mathbf{d}, e_\sigma, v$, and $\mathbf{e_T}$ are as shown in the block diagram in Figure 0.8 where ω_E is the characteristic polynomial of the estimator matrix A_E [9]. In developing this diagram we've represented the system defined by (0.12) as two separate subsystems, namely

$$\begin{aligned} \dot{x}_1 &= A_{\sigma\sigma}x_1 + h_\sigma e_\sigma \\ v &= f_{\sigma\sigma}x_1 + g_\sigma e_\sigma \end{aligned} \qquad \begin{aligned} \dot{x}_2 &= A_{\sigma\sigma}x_2 + h_\sigma e_\sigma \\ \mathbf{e_T} &= e_\sigma - \bar{c}_\sigma x_2 \end{aligned}$$

where $x_1 = x_2 = x$. Note that the signal in the block diagram labeled $\mathbf{b}$, will tend to zero if $\mathbf{d}$ is constant because of the zero at $s = 0$ in the numerator of the transfer function in the block driven by $\mathbf{d}$.

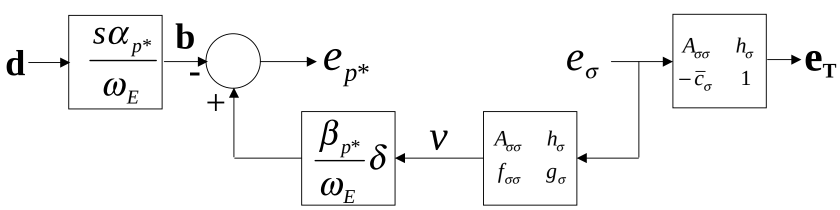

Fig. 0.8. Block Diagram I

Let us note that each of the five blocks in Figure 0.8 represents an exponentially stable linear system. It is convenient at this point to introduce certain "system gains" associated with these blocks. In particular, let us define for $p \in \mathcal{P}$

$$\mathfrak{a}_p \triangleq \left\| \frac{s\alpha_p}{\omega_E} \right\|, \qquad \mathfrak{b}_p \triangleq \sqrt{2}\left\| \frac{\beta_p}{\omega_E} \right\| \left\{ \sup_{\sigma \in \mathbb{S}} \left\| \begin{matrix} A_{\sigma\sigma} & h_\sigma \\ f_{\sigma\sigma} & g_\sigma \end{matrix} \right\| \right\}, \qquad \mathfrak{c} \triangleq \sqrt{2} \sup_{\sigma \in \mathbb{S}} \left\| \begin{matrix} A_{\sigma\sigma} & h_\sigma \\ -\bar{c}_\sigma & 1 \end{matrix} \right\|$$

where, as defined earlier, $\| \cdot \|$ is the shifted infinity norm and $\mathbb{S}$ is the set of all admissible switching signals. In the light of Figure 0.8, it is easy to see that

$$||\mathbf{e_T}||_{\{0,\ t\}} \leq \frac{\mathfrak{c}}{\sqrt{2}}||e_\sigma||_{\{0,\ t\}}, \quad t \geq 0, \tag{0.13}$$

$$||e_{p^*}||_{\{0,\ t\}} \leq \epsilon_{p^*}\frac{\mathfrak{b}_{p^*}}{\sqrt{2}}||e_\sigma||_{\{0,\ t\}} + ||\mathbf{b}||_{\{0,\ t\}}, \quad t \geq 0, \tag{0.14}$$

and

$$||\mathbf{b}||_{\{0,\ t\}} \leq \mathfrak{a}_{p^*}||\mathbf{d}||_{\{0,\ t\}} \quad t \geq 0 \tag{0.15}$$

where ϵ_{p^*} is the norm bound on δ. The inequality in (0.13) bounds the norm of $\mathbf{e_T}$ in terms of the norm of e_σ whereas (0.14) and (0.15) bound the norm of e_{p^*} in terms of the norms of e_σ and $\mathbf{d}$. To develop a bound for the norm of $\mathbf{e_T}$ in terms of the norms of $\mathbf{d}$, it would therefore be enough to establish a bound for the norm of e_σ in terms of the norm of e_{p^*}. As a first step toward this end, we shall make use of the following result which is a direct consequence of dwell time switching.

0.4.4 Dwell-Time Switching

Lemma 0.4.1. *Suppose that $\mathcal{P}$ contains $m > 0$ elements, that W is generated by (0.7), that the π_p, $p \in \mathcal{P}$, are defined by (0.6) and (0.8), that $W(0) = 0$, and that σ is the response of Σ_D to W. For each fixed time $T > 0$, there exists a piecewise-constant function $\psi_T : [0, \infty) \to \{0, 1\}$ such that*

$$\int_0^\infty \psi_T(t)dt \leq m(\tau_D + \tau_C) \tag{0.16}$$

and

$$||(1 - \psi_T)e_\sigma + \psi_T e_q||_{\{0,T\}} \leq \sqrt{m}||e_q||_{\{0,T\}}, \quad q \in \mathcal{P} \tag{0.17}$$

This lemma is proved in the appendix.

0.4.5 Block Diagram II

Let us fix $T > 0$. In view of (0.17) there is a piecewise constant signal $\psi_T : [0, \infty) \to \{0, 1\}$ satisfying (0.16) such that

$$||(1 - \psi_T)e_\sigma + \psi_T e_{p^*}||_{\{0,T\}} \leq \sqrt{m}||e_{p^*}||_{\{0,T\}} \tag{0.18}$$

But as noted before, what we need is a bound for the norm of e_σ, not $(1 - \psi_T)e_\sigma + \psi_T e_{p^*}$. To get around this, we first note from (0.12) that for $q \in \mathcal{P}$

$$e_\sigma - e_q = c_{\sigma q}x \qquad \dot{x} = A_{\sigma\sigma}x + h_\sigma e_\sigma \tag{0.19}$$

Next consider the block diagram in Figure 0.9 which depicts this sub-system with $q \stackrel{\Delta}{=} p^*$ together with an additional block and summing junctions representing the formulas

$$e_\sigma = \psi_T e_\sigma + (1 - \psi_T)e_\sigma \qquad \text{and} \qquad e_\sigma = e_{p^*} + e_\sigma - e_{p^*}$$

Let us define for $q \in \mathcal{P}$

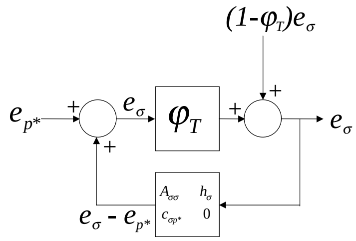

Fig. 0.9. Block Diagram II

$$\boxed{\mathfrak{v}_q \triangleq \sup_{\sigma\in\mathbb{S}}\sup_{t\geq 0}\int_0^t |w_q(t,\tau)e^{\lambda(t-\tau)}|^2 d\tau}$$

where $w_q(t,\tau) \triangleq c_{\sigma(t)q}\Phi(t,\tau)h_{\sigma(\tau)}$ and $\Phi(t,\tau)$ is the state transition matrix of $A_{\sigma\sigma}$. Note that each $\mathfrak{v}_q$ is finite because of the Slow Switching Assumption. Using Cauchy-Schwartz it can easily be shown with $\mathfrak{v}_q$ so defined that

$$||\psi_T(w_q \circ e_\sigma)||_{\{0,\ t\}} \leq \sqrt{\mathfrak{v}_q \int_0^t \psi_T^2 ||e_\sigma||^2_{\{0,\ \mu\}} d\mu}, \qquad t \geq 0 \tag{0.20}$$

where $w_q \circ e_\sigma$ is the zero initial state output response of (0.19).

From Figure 0.9 it is clear that

$$e_\sigma = \psi_T(e_{p^*} + w_{p^*} \circ e_\sigma) + (1 - \psi_T)e_\sigma$$

Rearranging terms and taking norms we thus obtain

$$||e_\sigma||_{\{0,\ t\}} \leq ||(1-\psi_T)e_\sigma + \psi_T e_{p^*}||_{\{0,\ t\}} + ||\psi_T(w_{p^*} \circ e_\sigma)||_{\{0,\ t\}}, \quad t \geq 0$$

Moreover $||(1-\psi_T)e_\sigma + e_{p^*}||_{\{0,t\}} \leq ||(1-\psi_T)e_\sigma + e_{p^*}||_{\{0,T\}}$, $t \in [0,\ T]$. Using (0.18) we thus get

$$||e_\sigma||_{\{0,\ t\}} \leq \sqrt{m}||e_{p^*}||_{\{0,T\}} + ||\psi_T(w_{p^*} \circ e_\sigma)||_{\{0,\ t\}}, \quad t \geq 0$$

Taking squares

$$||e_\sigma||^2_{\{0,\ t\}} \leq 2m||e_{p^*}||^2_{\{0,\ T\}} + 2||\psi_T(w_{p^*} \circ e_\sigma)||^2_{\{0,\ t\}}, \quad 0 \leq t \leq T$$

Using (0.20) with $q = p^*$

$$||e_\sigma||^2_{\{0,\ t\}} \leq 2m||e_{p^*}||^2_{\{0,\ T\}} + 2\mathfrak{v}_{p^*}\int_0^t \psi_T^2 ||e_\sigma)||^2_{\{0,\ \mu\}} d\mu, \quad 0 \leq t \leq T$$

Hence by the Bellman-Gronwall Lemma

$$||e_\sigma||^2_{\{0,\ T\}} \leq 2m||e_{p^*}||^2_{\{0,\ T\}} e^{2\mathfrak{v}_{p^*}\int_0^T \psi_T^2 dt}$$

From this, (0.16), and the fact that $\psi_T^2 = \psi_T$, we arrive finally at an expression for the norm of e_σ in terms of e_{p^*}, namely

$$||e_\sigma||_{\{0,\ T\}} \leq \sqrt{2m}e^{\mathfrak{v}_{p^*} m(\tau_D + \tau_C)}||e_{p^*}||_{\{0,\ T\}}, \quad T \geq 0 \tag{0.21}$$

0.4.6 Stability Margin

Examination of (0.21) and (0.14) reveals that if ϵ_{p^*} satisfies the small gain condition

$$\boxed{\epsilon_{p^*} < \frac{e^{-\mathfrak{v}_{p^*} m(\tau_D+\tau_C)}}{\mathfrak{b}_{p^*}\sqrt{m}}} \tag{0.22}$$

then (0.21) and (0.14) can be combined to give

$$||e_\sigma||_{\{0,\ T\}} \leq \frac{\sqrt{2}}{\frac{e^{-\mathfrak{v}_{p^*} m(\tau_D+\tau_C)}}{\sqrt{m}} - \epsilon_{p^*}\mathfrak{b}_{p^*}} ||\mathbf{b}||_{\{0,\ T\}}, \quad T \geq 0 \tag{0.23}$$

The inequality in (0.22) provides an explicit bound for the allowable process dynamics.

0.4.7 Global Boundedness

The global boundedness condition of Theorem 0.3.1 can now easily be justified as follows. Suppose $\mathbf{d}$ is bounded on $[0,\ \infty)$. Then so must be $e^{-\lambda t}||\mathbf{b}||_{\{0,\ t\}}$. Hence by (0.23), $e^{-\lambda t}||e_\sigma||_{\{0,\ t\}}$ must be bounded on $[0,\ \infty)$ as well. This, the differential equation for x in (0.12), and the exponential stability of $\lambda I + A_{\sigma\sigma}$ then imply that x is also bounded on $[0,\ \infty)$. In view of (0.10) and (0.11), x_E and x_C must also be bounded. Next recall that the zeros of ω_E {i.e., the eigenvalues of A_E} have negative real parts less than $-\lambda$, and that the transfer function $\frac{\beta_{p^*}}{\omega_E}\delta$ in Figure 0.8 is strictly proper. From these observations and the block diagram in Figure 0.8 one readily concludes that e_{p^*} is bounded on $[0, \infty)$. Hence from the formulas in (0.12) for e_σ, v and $\mathbf{e_T}$ one concludes that these signals are also bounded. In view of (0.3), y must be bounded. Thus W must be bounded because of (0.7). Finally note that u must be bounded because of the boundedness of $y - \mathbf{n}$ and v and because of the observability of the cascade interconnection of (0.2) with any minimal realization of Σ_P. This, in essence, proves Claim 1 of Theorem 0.3.1.

0.4.8 Convergence

Now suppose that $\mathbf{d}$ is a constant. Examination of Figure 0.8 reveals that $\mathbf{b}$ must tend to zero as fast as $e^{-\lambda t}$ because of the zero at $s = 0$ in the numerator of the transfer function from $\mathbf{d}$ to $\mathbf{b}$. This implies that $||\mathbf{b}||_{\{0,\ \infty\}} < \infty$. Therefore $||e_\sigma||_{\{0,\ \infty\}} < \infty$ because of (0.23). Hence e_σ must tend to zero as fast as $e^{-\lambda t}$. So therefore must x because of the differential equation for x in (0.12). In view of (0.10) and (0.11) $\bar{x}_E$ and x_C must tend to zero as well. From Block Diagram I in Figure 0.8 it now can be seen that e_{p^*} tends to zero. Hence from the formulas in (0.12) for e_σ, v and $\mathbf{e_T}$ one concludes that these signals must tend to zero as well. In view of (0.3), y must tent to r . Thus W must approach a finite limit because of (0.7). Finally note that u tend to a finite limit because y and v do and because of the observability of the cascade interconnection of (0.2) with any minimal realization of Σ_P. This, in essence, proves Claim 2 of Theorem 0.3.1.

0.4.9 A Bound on the Disturbance - to - Tracking - Error Gain

By combining the inequalities in (0.13), (0.15) and (0.23) we obtain an inequality of the form

$$||\mathbf{e_T}||_{\{0,\ T\}} \leq \mathfrak{g}_{p^*}||\mathbf{d}||_{\{0,\ T\}}, \quad T \geq 0$$

where

$$\boxed{\mathfrak{g}_{p^*} = \frac{\mathfrak{c}\mathfrak{a}_{p^*}}{\frac{e^{-\mathfrak{v}_{p^*} m(\tau_D+\tau_C)}}{\sqrt{m}} - \epsilon_{p^*}\mathfrak{b}_{p^*}}} \tag{0.24}$$

Thus $\mathfrak{g}_{p^*}$ bounds from above the overall system's disturbance - to - tracking - error gain.

0.5 Concluding Remarks

The formula for $\mathfrak{g}_{p^*}$ in (0.24) and the stability margin bound in (0.22) are probably the most explicit discovered so far for an estimator-based adaptive control system with the properties outlined in Theorem 0.3.1. We believe that even simpler expressions than these can be found for the system under consideration. For example, it is likely that instead of (0.22), it will suffice to bound ϵ_{p^*} by $\frac{1}{\mathfrak{b}_{p^*}\sqrt{m}}$. Under certain conditions, it also possible to derive useful relationships between system gains and the the shifted infinity norms of the transfer functions of the constant linear systems being switched. For example, for τ_D sufficiently large, any strict upper bound on the family $\{\sqrt{2}\|1 - \bar{c}_p(sI - A_{pp})^{-1}h_p\| \ :\ p \in \mathcal{P}\}$ is an upper bound on $\mathfrak{c}$ [14]. Results such as these suggest that a bona fide, input-output performance theory for adaptive control may be within our reach.

0.6 Appendix

In the sequel, σ is a fixed switching signal, $t_0 \stackrel{\Delta}{=} 0$, t_i denotes the ith time at which σ switches and p_i is the value of σ on $[t_{i-1}, t_i)$; if σ switches at most $n < \infty$ times then $t_{n+1} \stackrel{\Delta}{=} \infty$ and p_{n+1} denotes σ's value on $[t_n, \infty)$. Any time X takes on the current value of W is called a *sample time*. We use the notation $\lfloor t \rfloor$ to denote the sample time just preceding time t, if $t > \tau_D - \tau_C$, and the number zero otherwise. Thus, for example, $\lfloor t_0 \rfloor = 0$ and $\lfloor t_i \rfloor = t_i - \tau_C,\ i > 0$.

To prove Lemma 0.4.1 will need the following algebraic fact

Lemma 0.6.1. *For all* $\mu_i \in [0,1],\ i \in \{1, 2, \ldots, m\}$

$$\sum_{i=1}^{m}(1-\mu_i) \leq (m-1) + \prod_{i=1}^{m}(1-\mu_i) \tag{0.25}$$

Proof of Lemma 0.6.1: Set $x_i = 1 - \mu_i,\ i \in \{1, 2, \ldots, \}$. It is enough to show that for $x_i \in [0,1],\ i \in \{1, 2, \ldots, \}$

$$\sum_{i=1}^{j} x_i \leq (j-1) + \prod_{i=1}^{j} x_j \tag{0.26}$$

for $j \in \{1, 2, \ldots, m\}$. Clearly (0.26) is true if $j = 1$. Suppose therefore that for some $k > 0$, (0.26) holds for $j \in \{1, 2, \ldots, k\}$. Then

$$\begin{aligned}
\sum_{i=1}^{k+1} x_i &= x_{k+1} + \sum_{i=1}^{k} x_i \\
&\leq x_{k+1} + (k-1) + \prod_{i=1}^{k} x_i \\
&\leq (1 - x_{k+1}) \left(1 - \prod_{i=1}^{k} x_i \right) + x_{k+1} + (k-1) + \prod_{i=1}^{k} x_i \\
&= k + \prod_{i=1}^{k+1} x_i
\end{aligned}$$

By induction, (0.26) thus holds for $j \in \{1, 2, \ldots, m\}$. ■

Proof of Lemma 0.4.1: Let $\mathcal{P}_T$ be the image of $[0, T]$ under σ. Let k be that integer for which $T \in [t_{k-1},\ t_k)$. For each $p \in \mathcal{P}_T$, let $\mathcal{I}_p$ denote the set of nonnegative integers i such that $\sigma = p$ for $t \in [t_{i-1}, t_i)$. Let j_p denote the largest integer in $\mathcal{I}_p$. Note that $j_{p_k} = k$.

For each $i \in \{1, 2, \ldots, k\}$ define

$$\bar{t}_i = \begin{cases} t_i & \text{if } i < k \\ T & \text{if } i = k \end{cases}$$

The definition of dwell-time switching then implies that for $p \in \mathcal{P}_T$,

$$\begin{aligned}
\pi_p(\lfloor t_{i-1} \rfloor) &\leq \pi_q(\lfloor t_{i-1} \rfloor), \qquad \forall q \in \mathcal{P}, i \in \mathcal{I}_p \\
\pi_p(\lfloor \bar{t}_i \rfloor - \tau_C) &\leq \pi_q(\lfloor \bar{t}_i \rfloor - \tau_C), \quad \forall q \in \mathcal{P}, i \in \mathcal{I}_p \ \text{ if } \bar{t}_i - t_{i-1} > \tau_D
\end{aligned}$$

Setting $i \triangleq j_p$ and using the fact that $e^{2\lambda t}\pi_p(t) = ||e_p||^2_{\{0,t\}}, \quad p \in \mathcal{P},\ t \geq 0$, we obtain the expressions

$$\left.\begin{aligned}
||e_p||^2_{\{0,\ \lfloor t_{j_p - 1} \rfloor\}} &\leq ||e_q||^2_{\{0,\ \lfloor t_{j_p - 1} \rfloor\}}, \qquad \forall q \in \mathcal{P} \\
||e_p||^2_{\{0,\ \lfloor \bar{t}_{j_p} \rfloor - \tau_C\}} &\leq ||e_q||^2_{\{0,\ \lfloor \bar{t}_{j_p} \rfloor - \tau_C\}}, \quad \forall q \in \mathcal{P}, \text{ if } \bar{t}_{j_p} - t_{j_p - 1} > \tau_D
\end{aligned}\right\} \tag{0.27}$$

For each $p \in \mathcal{P}_T$, let $\phi_p : [0, \infty) \rightarrow \{0, 1\}$ be that piecewise-constant signal which is zero everywhere except on the interval

$$[\lfloor t_{j_p - 1} \rfloor,\ \bar{t}_{j_p}), \qquad \text{if } \ \bar{t}_{j_p} - t_{j_p - 1} \leq \tau_D$$

or

$$[\lfloor \bar{t}_{j_p} \rfloor - \tau_C,\ \bar{t}_{j_p}), \ \text{ if } \ \bar{t}_{j_p} - t_{j_p - 1} > \tau_D$$

In either case ϕ_p has support no greater than $\tau_D + \tau_C$ and is idempotent {i.e., $\phi_p^2 = \phi_p$}. It follows that if

$$\psi_T \triangleq 1 - \prod_{p \in \mathcal{P}_T} (1 - \phi_p), \tag{0.28}$$

then ψ_T is also idempotent and has support no greater than $m(\tau_D + \tau_C)$. In view of the latter property, (0.16) must be true.

The definitions of the ϕ_p imply that for $p \in \mathcal{P}_T$ and $l \in \mathcal{P}$

$$||(1-\phi_p)e_l||^2_{\{0,\bar{t}_{j_p}\}} = \begin{cases} ||(1-\phi_p)e_l||^2_{\{0,\ \lfloor t_{j_p-1}\rfloor\}} & \text{if } \bar{t}_{j_p} - t_{j_p-1} \leq \tau_D \\ ||(1-\phi_p)e_l||^2_{\{0,\ \lfloor \bar{t}_{j_p}\rfloor - \tau_C\}} & \text{if } \bar{t}_{j_p} - t_{j_p-1} > \tau_D \end{cases}$$

From this and (0.27) we obtain for all $q \in \mathcal{P}$

$$||(1-\phi_p)e_p||^2_{\{0,\bar{t}_{j_p}\}} = ||e_p||^2_{\{0,\ \lfloor t_{j_p-1}\rfloor\}} \leq ||e_q||^2_{\{0,\ \lfloor t_{j_p-1}\rfloor\}} = ||(1-\phi_p)e_q||^2_{\{0,\bar{t}_{j_p}\}}$$

if $\bar{t}_{j_p} - t_{j_p-1} \leq \tau_D$ and

$$||(1-\phi_p)e_p||^2_{\{0,\bar{t}_{j_p}\}} = ||e_p||^2_{\{0,\ \lfloor \bar{t}_{j_p} - \tau_C\rfloor\}} \leq ||e_q||^2_{\{0,\ \lfloor \bar{t}_{j_p} - \tau_C\rfloor\}} = ||(1-\phi_p)e_q||^2_{\{0,\bar{t}_{j_p}\}}$$

if $\bar{t}_{j_p} - t_{j_p-1} > \tau_D$. From this and the fact that

$$||(1-\phi_p)e_q||^2_{\{0,\bar{t}_{j_p}\}} \leq ||(1-\phi_p)e_q||^2_{\{0,T\}},\ q \in \mathcal{P},\ p \in \mathcal{P}_P$$

there follows

$$||(1-\phi_p)e_p||^2_{\{0,\bar{t}_{j_p}\}} \leq ||(1-\phi_p)e_q||^2_{\{0,T\}}, \quad \forall p \in \mathcal{P}_T,\ \ q \in \mathcal{P} \tag{0.29}$$

Now

$$||(1-\psi_T)e_\sigma||^2_{\{0,T\}} = \sum_{p\in\mathcal{P}_T}\sum_{i\in\mathcal{I}_p} ||(1-\psi_T)e_p||^2_{\{t_{i-1},\bar{t}_i\}} \leq \sum_{p\in\mathcal{P}_T} ||(1-\psi_T)e_p||^2_{\{0,\bar{t}_{j_p}\}} \tag{0.30}$$

In view of (0.28) we can write

$$\sum_{p\in\mathcal{P}_T} ||(1-\psi_T)e_p||^2_{\{0,\bar{t}_{j_p}\}} = \sum_{p\in\mathcal{P}_T} \left\| \left\{ \prod_{l\in\mathcal{P}_T} (1-\phi_l) \right\} e_p \right\|^2_{\{0,\bar{t}_{j_p}\}} \tag{0.31}$$

But

$$\sum_{p\in\mathcal{P}_T} \left\| \left\{ \prod_{l\in\mathcal{P}_T} (1-\phi_l) \right\} e_p \right\|^2_{\{0,\bar{t}_{j_p}\}} \leq \sum_{p\in\mathcal{P}_T} \|(1-\phi_p)e_p\|^2_{\{0,\bar{t}_{j_p}\}}$$

From this, (0.29), (0.30), and (0.31) it follows that

$$||(1-\psi_T)e_\sigma||^2_{\{0,T\}} \leq \sum_{p\in\mathcal{P}_T} ||(1-\phi_p)e_q||^2_{\{0,T\}}, \quad \forall\ \ q \in \mathcal{P}$$

Thus for $q \in \mathcal{P}$

$$\begin{aligned} ||(1-\psi_T)e_\sigma||^2_{\{0,T\}} &\leq \sum_{p\in\mathcal{P}_T} \int_0^T \{e_q e^{\lambda t}\}^2 (1-\phi_p)^2 dt \\ &= \int_0^T \{e_q e^{\lambda t}\}^2 \left\{ \sum_{p\in\mathcal{P}_T} (1-\phi_p)^2 \right\} dt \\ &= \int_0^T \{e_q e^{\lambda t}\}^2 \left\{ \sum_{p\in\mathcal{P}_T} (1-\phi_p) \right\} dt \end{aligned}$$

This, Lemma 0.6.1 and (0.28) imply that

$$\begin{aligned} ||(1-\psi_T)e_\sigma||^2_{\{0,T\}} &\leq \int_0^T \{e_q e^{\lambda t}\}^2 \left\{ m - 1 + \prod_{p \in \mathcal{P}_T} (1-\phi_p) \right\} dt \\ &= \int_0^T \{e_q e^{\lambda t}\}^2 \{m - \psi_T\} dt \\ &= \int_0^T \{e_q e^{\lambda t}\}^2 \{m - \psi_T^2\} dt \end{aligned}$$

Hence

$$||(1-\psi_T)e_\sigma||^2_{\{0,T\}} \leq m||e_q||^2_{\{0,T\}} - ||\psi_T e_q||^2_{\{0,T\}} \tag{0.32}$$

Now

$$||(1-\psi_T)e_\sigma||^2_{\{0,T\}} + ||\psi_T e_q||^2_{\{0,T\}} = ||(1-\psi_T)e_\sigma + \psi_T e_q||^2_{\{0,T\}}$$

because $\psi_T(1-\psi_T) = 0$. From this and (0.32) it follows that

$$||(1-\psi_T)e_\sigma + \psi_T e_q||^2_{\{0,T\}} \leq m||e_q||^2_{\{0,T\}}$$

and thus that (0.17) is true. ■

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