

# Coordination of Groups of Mobile Autonomous Agents Using Nearest Neighbor Rules\*

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March 24, 2002

## Abstract

In a recent *Physical Review Letter*, Vicsek et. al. propose a simple but compelling discrete-time model of  $n$  autonomous agents {i.e., points or particles} all moving in the plane with the same speed but with different headings. Each agent's heading is updated using a local rule based on the average of its own heading plus the headings of its "neighbors." In their paper, Vicsek et. al. provide simulation results which demonstrate that the nearest neighbor rule they are studying can cause all agents to eventually move in the same direction despite the absence of centralized coordination and despite the fact that each agent's set of nearest neighbors change with time as the system evolves. This paper provides a theoretical explanation for this observed behavior. In addition, convergence results are derived for several other similarly inspired models. The Vicsek model proves to be a graphic example of a switched linear system which is stable, but for which there does not exist a common quadratic Lyapunov function.

## 1 Introduction

In a recent paper [1], Vicsek et. al. propose a simple but compelling discrete-time model of  $n$  autonomous agents {i.e., points or particles} all moving in the plane with the same speed but with different headings. Each agent's heading is updated using a local rule based on the average of its own heading plus the headings of its "neighbors." Agent  $i$ 's *neighbors* at time  $t$ , are those agents which are either in or on a circle of pre-specified radius  $r$  centered at agent  $i$ 's current position. The Vicsek model turns out to be a special version of a model introduced previously by Reynolds [2] for simulating visually satisfying flocking and schooling behaviors for the animation industry. In their paper, Vicsek et. al. provide a variety of interesting simulation results which demonstrate

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\*This research was supported by DARPA under its SEC program and by the NSF under its KDI/LIS initiative.

that the nearest neighbor rule they are studying can cause all agents to eventually move in the same direction despite the absence of centralized coordination and despite the fact that each agent's set of nearest neighbors change with time as the system evolves. In this paper we provide a theoretical explanation for this observed behavior.

There is a large and growing literature concerned with the coordination of groups of mobile autonomous agents. Included here is the work of Czirok *et al.* [3] who propose one-dimensional models which exhibit the same type of behavior as Vicsek's. In [4, 5], Toner and Tu construct a continuous "hydrodynamic" model of the group of agents, while other authors such as Mikhailov and Zanette [6] consider the behavior of populations of self propelled particles with long range interactions. Schenk *et al.* determined interactions between individual self-propelled spots from underlying reaction-diffusion equation [7].

In addition to these modelling and simulation studies, research papers focusing on the detailed mathematical analysis of emergent behaviors are beginning to appear. For example, Leonard *et al.* [8] use potential function theory to understand flocking behavior while Fax and Murray [9] and Desai *et al.* [10] employ graph theoretic techniques for the same purpose. The one feature which sharply distinguishes previously analyzed models from those considered here is that the latter undergo changes in nearest neighbors over time, whereas the former do not. This feature is inherent in the Vicsek model and in the other models we consider. To analyze such models, it proves useful to appeal to well-known results [11, 12] characterizing the convergence of infinite products of certain types of non-negative matrices. The study of infinite matrix products is ongoing [13, 14, 15, 16, 17, 18] and is undoubtedly producing results which will find application in the theoretical study of emergent behaviors.

Vicsek's model is set up in Section 2 as a system of  $n$  simultaneous, one-dimensional recursion equations, one for each agent. A family of simple graphs on  $n$  vertices is then introduced to characterize all possible neighbor relationships. Doing this makes it possible to represent the Vicsek model as an  $n$ -dimensional switched linear system whose switching signal takes values in the set of indices which parameterize the family of graphs. The matrices which are switched within the system turn out to be non-negative with special structural properties. By exploiting these properties and making use of a classical convergence result due to Wolfowitz [11], we prove that all  $n$  agents' headings converge to a common steady state heading provided the  $n$  agents are all "linked together" via their neighbors with sufficient frequency as the system evolves. The model under consideration proves to be a graphic example of a switched linear system which is stable, but for which there does not exist a common quadratic Lyapunov function.

In Section 2.2 we define the notion of an average heading vector in terms of graph Laplacians [19] and we shown how this idea leads naturally to the Vicsek model as well as to other decentralized control models which might be used for the same purposes. We propose one such model which assumes each agent knows an upper bound on the number of agents in the group, and we explain why this model has the convergence properties similar to Vicsek's.

In Section 3 we consider a modified version of Vicsek's discrete-time system consisting of the same group of  $n$  agents, plus one additional agent, labelled 0, which acts as the group's leader. Agent 0 moves at the same constant speed as its  $n$  followers but with a fixed heading  $\theta_0$ . The  $i$ th follower updates its heading just as in the Vicsek model, using the average of its own heading plus the headings of its neighbors. For this system, each follower's set of neighbors can also include

the leader and does so whenever the leader is within the follower’s neighborhood defining circle of radius  $r$ . We prove that the headings of all  $n$  agents must converge to the leader’s provided all  $n$  agents plus their leader are linked together via their neighbors frequently enough as the system evolves. Finally we develop a continuous-time analog of this system and prove under condition more mild than imposed in the discrete-time case, that the headings of all  $n$  agents again converge to the heading of the group’s leader.

## 2 Leaderless Coordination

The system studied by Vicsek et. al. in [1] consists of  $n$  autonomous agents {e.g., points or particles}, labelled 1 through  $n$ , all moving in the plane with the same speed but with different headings<sup>1</sup>. Each agent’s heading is updated using a simple local rule based on the average of its own heading plus the headings of its “neighbors.” Agent  $i$ ’s *neighbors* at time  $t$ , are those agents which are either in or on a circle of pre-specified radius  $r$  centered at agent  $i$ ’s current position. In the sequel  $\mathcal{N}_i(t)$  denotes the set of labels of those agents which are neighbors of agent  $i$  at time  $t$ . Agent  $i$ ’s heading, written  $\theta_i$ , evolves in discrete-time in accordance with a model of the form

$$\theta_i(t+1) = \langle \theta_i(t) \rangle_r \quad (1)$$

where  $t$  is a discrete-time index taking values in the non-negative integers  $\{0, 1, 2, \dots\}$ , and  $\langle \theta_i(t) \rangle_r$  is the average of the headings of agent  $i$  and agent  $i$ ’s neighbors at time  $t$ ; that is

$$\langle \theta_i(t) \rangle_r = \frac{1}{1 + n_i(t)} \left( \theta_i(t) + \sum_{j \in \mathcal{N}_i(t)} \theta_j(t) \right) \quad (2)$$

where  $n_i(t)$  is the number of neighbors of agent  $i$  at time  $t$ .

The explicit form of the update equations determined by (1) and (2) depends on the relationships between neighbors which exist at time  $t$ . These relationships can be conveniently described by a simple, undirected graph with vertex set  $\{1, 2, \dots, n\}$  which is defined so that  $(i, j)$  is one of the graph’s edges just in case agents  $i$  and  $j$  are neighbors. Since the relationships between neighbors can change over time, so can the graph which describes them. To account for this we will need to consider all possible such graphs. In the sequel we use the symbol  $\mathcal{P}$  to denote a suitably defined set, indexing the class of all simple graphs  $\mathbb{G}_p$  defined on  $n$  vertices.

The set of agent heading update rules defined by (1) and (2), can be written in state form. Toward this end, for each  $p \in \mathcal{P}$ , define

$$F_p = (I + D_p)^{-1}(A_p + I) \quad (3)$$

where  $A_p$  is the adjacency matrix<sup>2</sup> of graph  $\mathbb{G}_p$  and  $D_p$  the diagonal matrix whose  $i$ th diagonal element is the valence of vertex  $i$  within the graph. Then

$$\theta(t+1) = F_{\sigma(t)}\theta(t), \quad t \in \{0, 1, 2, \dots\} \quad (4)$$

<sup>1</sup>The Vicsek system also includes noise input signals which we ignore in this paper.

<sup>2</sup>The adjacency matrix of a simple graph on  $n$  vertices is an  $n \times n$  matrix of whose  $ij$ th entry is 1 if  $(i, j)$  is one of the graph’s edges and 0 if it is not.

where  $\theta$  is the heading vector  $\theta = [\theta_1 \ \theta_2 \ \dots \ \theta_n]'$  and  $\sigma : \{0, 1, \dots\} \rightarrow \mathcal{P}$  is a switching signal whose value at time  $t$ , is the index of the graph representing the agent system's neighbor relationships at time  $t$ . A complete description of this system would have to include a model which explains how  $\sigma$  changes over time as a function of the positions of the  $n$  agents in the plane. While such a model is easy to derive and is essential for simulation purposes, it would be difficult to take into account in a convergence analysis. To avoid this difficulty, we shall adopt a more conservative approach which ignores how  $\sigma$  depends on the agent positions in the plane and assumes instead that  $\sigma$  might be any switching signal in some suitably defined set of interest.

Our goal is to show for a large class of switching signals and for any initial set of agent headings that the headings of all  $n$  agents will converge to the same steady state value  $\theta_{ss}$ . Convergence of the  $\theta_i$  to  $\theta_{ss}$  is equivalent to the state vector  $\theta$  converging to a vector of the form  $\theta_{ss}\mathbf{1}$  where  $\mathbf{1} \triangleq [1 \ 1 \ \dots \ 1]_{n \times 1}'$ . Naturally there are situations when convergence to a common heading cannot occur. The most obvious of these is when one or more agents starts so far away from the rest that it never acquires any neighbors. Mathematically this would mean that the values of  $\sigma$  along such a trajectory would be such that  $\mathbb{G}_{\sigma(t)}$  is never a connected graph. This situation is likely to be encountered when  $r$  is very small. At the other extreme, which is likely when  $r$  is very large, all agents might remain neighbors of all others for all time. In this case,  $\sigma$  would remain fixed along such a trajectory at that value in  $p \in \mathcal{P}$  for which  $\mathbb{G}_p$  is a complete graph. Convergence of  $\theta$  to  $\theta_{ss}\mathbf{1}$  can easily be established in this special case because with  $\sigma$  so fixed, (4) is a linear, time-invariant, discrete-time system. The situation of perhaps the greatest interest is between these two extremes where  $\mathbb{G}_{\sigma(t)}$  might be disconnected only some time and connected but not necessarily complete the rest of the time. Establishing convergence in this case is challenging because  $\sigma$  changes with time and (4) is not time-invariant. It is this case which we intend to study. Towards this end, we denote by  $\mathcal{Q}$  the subset of  $\mathcal{P}$  consisting of the indices of the connected graphs in  $\{\mathbb{G}_p : p \in \mathcal{P}\}$ . Our first result establishes the convergence of  $\theta$  for the case when  $\sigma$  takes values only in  $\mathcal{Q}$ .

**Theorem 1** *Let  $\theta(0)$  be fixed and let  $\sigma : \{0, 1, 2, \dots\} \rightarrow \mathcal{P}$  be a switching signal satisfying  $\sigma(t) \in \mathcal{Q}$ ,  $t \in \{0, 1, \dots\}$ . Then*

$$\lim_{t \rightarrow \infty} \theta(t) = \theta_{ss}\mathbf{1} \tag{5}$$

where  $\theta_{ss}$  is a number depending only on  $\theta(0)$  and  $\sigma$ .

It is natural to say that the  $n$  agents are *linked* together by their neighbors whenever the graph  $\mathbb{G}_p$  defining their neighbor relations is connected. Theorem 1 says that convergence of all agent's headings to a common heading is for certain provided all  $n$  agents are always linked together in this way. Of course there is no guarantee that along a specific trajectory the  $n$  agents will be so linked. Perhaps a more likely situation, at least when  $r$  is not too small, is when the agents are linked together sufficiently often. With proper interpretation of "sufficiently often" convergence to a common heading is for certain in this case too.

**Theorem 2** *Let  $\theta(0)$  be fixed and let  $\sigma : \{0, 1, 2, \dots\} \rightarrow \mathcal{P}$  be a switching signal for which there exists a positive integer  $T$  large enough so that  $\sigma(t) \in \mathcal{Q}$  for at least one value of  $t$  in each time-interval of length  $T$ . Then*

$$\lim_{t \rightarrow \infty} \theta(t) = \theta_{ss}\mathbf{1} \tag{6}$$

where  $\theta_{ss}$  is a number depending only on  $\theta(0)$  and  $\sigma$ .

The hypotheses in Theorem 2 require  $\sigma$  to take a value in  $\mathcal{Q}$  at least once on each time interval of length  $T$ . Although no constraints are placed on  $T$  other than that it be finite, the constraint on  $\sigma$  is more restrictive than one might hope for. What one would prefer instead is to show that (6) holds for every switching signal  $\sigma$  for which there is an infinite subsequence of times  $t_1, t_2, \dots$  such that  $\sigma(t_i) \in \mathcal{Q}$ ,  $i \geq 0$ . Whether or not this is true remains to be seen.

Since Theorem 1 is obviously a consequence of Theorem 2, we need only develop a proof of the latter. To do this we will make use of certain structural properties of the  $F_p$ . As defined, each  $F_p$  is square and non-negative, where by a *non-negative* matrix is meant a matrix whose entries are all non-negative. Each  $F_p$  also has the property that its row sums all equal 1 {i.e.,  $F_p \mathbf{1} = \mathbf{1}$ }. Matrices with these two properties are called *stochastic* [20]. The  $F_p$  have the additional property that their diagonal elements are all non-zero. For the case when  $p \in \mathcal{Q}$  {i.e., when  $\mathbb{G}_p$  is connected}, it is known that  $(I + A_p)^m$  becomes a matrix with all positive entries for  $m$  sufficiently large [20]. It is easy to see that if  $(I + A_p)^m$  has all positive entries, then so does  $F_p^m$ . Such  $(I + A_p)$  and  $F_p$  are examples of “primitive matrices” where by a *primitive* matrix is meant any square, non-negative matrix  $M$  for which  $M^m$  is a matrix with all positive entries for  $m$  sufficiently large [20]. It is known [20] that among the  $n$  eigenvalues of a primitive matrix, there is exactly one with largest magnitude, that this eigenvalue is the only one possessing an eigenvector with all positive entries, and that the remaining  $n - 1$  eigenvalues are all strictly smaller in magnitude than the largest one. This means that for  $p \in \mathcal{Q}$ , 1 must be  $F_p$ ’s largest eigenvalue and all remaining eigenvalues must lie within the unit circle. As a consequence, each such  $F_p$  must have the property that  $\lim_{i \rightarrow \infty} F_p^i = \mathbf{1}c_p$  for some row vector  $c_p$ . Any stochastic matrices  $M$  for which  $\lim_{i \rightarrow \infty} M^i$  is a matrix of rank 1 is called *ergodic* [20]. Primitive stochastic matrices are thus ergodic matrices. To summarize, each  $F_p$  is a stochastic matrix with positive diagonal elements and if  $p \in \mathcal{Q}$  then  $F_p$  is also primitive and hence ergodic. The crucial convergence result upon which the proof of Theorem 2 depends is classical [11] and is as follows.

**Theorem 3 (Wolfowitz)** *Let  $M_1, M_2, \dots, M_m$  be a finite set of ergodic matrices with the property that for each sequence  $M_{i_1}, M_{i_2}, \dots, M_{i_j}$  of positive length, the matrix product  $M_{i_j} M_{i_{j-1}} \cdots M_{i_1}$  is ergodic. Then for each infinite sequence  $M_{i_1}, M_{i_2}, \dots$  there exists a row vector  $c$  such that*

$$\lim_{j \rightarrow \infty} M_{i_j} M_{i_{j-1}} \cdots M_{i_1} = \mathbf{1}c$$

In order to make use of Theorem 3, we need a few facts concerning products of the types of matrices we are considering. First we point out that the class of  $n \times n$  stochastic matrices with positive diagonal elements is closed under matrix multiplication. This is because the product of two non-negative matrices with positive diagonals is a matrix with the same properties and because the product of two stochastic matrices is stochastic. Second we will use the following key result<sup>3</sup>.

**Lemma 1** *Let  $A$  and  $B$  be two  $n \times n$  non-negative matrices. If the diagonal elements of  $A$  are positive and if  $B$  is primitive, then  $AB$  and  $BA$  are primitive as well.*

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<sup>3</sup>We are indebted to Marc Artzrouni, University of Pau, France for his help with this lemma’s proof.

The implications of Lemma 1 and the brief discussion which precedes it are (i) that the product of any finite number of  $F_p$  is stochastic with positive diagonal elements and (ii) that any such product is primitive provided it contains at least one matrix  $F_q$ , anywhere in the product, for which  $q \in \mathcal{Q}$ .

**Proof of Theorem 2<sup>4</sup>:** Let  $\Phi(t, t) = I$ ,  $t \geq 0$  and  $\Phi(t, \tau) \triangleq F_{\sigma(t-1)} \cdots F_{\sigma(1)} F_{\sigma(\tau)}$ ,  $t > \tau \geq 0$ . Clearly  $\theta(t) = \Phi(t, 0)\theta(0)$ . To complete the theorem's proof, it is therefore enough to show that

$$\lim_{t \rightarrow \infty} \Phi(t, 0) = \mathbf{1}c \quad (7)$$

for some row vector  $c$  since this would imply (6) with  $\theta_{ss} \triangleq c\theta(0)$ . For all  $j \geq 1$ , define  $M_j = F_{\sigma(jT-1)} F_{\sigma(jT-2)} \cdots F_{\sigma((j-1)T)}$ . The constraints on  $\sigma$  imply that each such matrix product  $M_j$ ,  $j \geq 1$  contains at least one matrix  $F_q$  which is primitive. It follows from this and the two properties of the  $F_p$  discussed just after Lemma 1, that any finite product composed of the  $M_j$  in any order must be stochastic and primitive, and therefore ergodic. Moreover the set of possible  $M_j$  must be finite because each  $M_j$  is a product of  $T$  matrices from  $\{F_p : p \in \mathcal{P}\}$  which is a finite set. But  $\Phi(iT, 0) = M_i M_{i-1} \cdots M_1$ . Therefore by Theorem 3,

$$\lim_{i \rightarrow \infty} \Phi(iT, 0) = \mathbf{1}c \quad (8)$$

Let  $\|\cdot\|$  denote any matrix norm on  $\mathbb{R}^{n \times n}$  which is sub-multiplicative {e.g., the infinity norm}. Let  $\epsilon$  be any number greater than 0. To establish (7) and thus complete the proof, it is enough to show that there is an integer  $m$  sufficiently large so that

$$\|\Phi(t, 0) - \mathbf{1}c\| < \epsilon, \quad \forall t \geq m \quad (9)$$

Toward this end, let  $\gamma$  denote the maximum of value of  $\|F_{p_1} F_{p_2} \cdots F_{p_i}\|$  over all sequences  $p_1, p_2, \dots, p_j$ ,  $p_i \in \mathcal{P}$ , of length less than  $T$ . In view of (8) it is possible to define  $m$  so that

$$\|\Phi(jT, 0) - \mathbf{1}c\| < \frac{\epsilon}{\gamma}, \quad \forall j \geq m \quad (10)$$

It remains to be shown that (9) holds with  $m$  so defined. For this, fix  $t \geq m$  and let  $j$  be the largest non-negative integer such that  $jT \leq t$ . Then  $\Phi(t, 0) = \Phi(t, jT)\Phi(jT, 0)$  and  $\Phi(t, jT)\mathbf{1} = \mathbf{1}$  so  $\Phi(t, 0) - \mathbf{1}c = \Phi(t, jT)(\Phi(jT, 0) - \mathbf{1}c)$ . Therefore

$$\|\Phi(t, 0) - \mathbf{1}c\| \leq \|\Phi(t, jT)\| \|(\Phi(jT, 0) - \mathbf{1}c)\| \quad (11)$$

Since  $\Phi(t, jT)$  is a product of  $F_p$  of length less than  $T$ ,  $\|\Phi(t, jT)\| \leq \gamma$ . From this, (10), and (11) it thus follows that  $\|\Phi(t, 0) - \mathbf{1}c\| < \epsilon$ . Therefore (9) holds. ■

To prove Lemma 1 we shall make use of the standard partial ordering  $\geq$  on  $n \times n$  non-negative matrices by writing  $B \geq A$  whenever  $B - A$  is a non-negative. Let us note that if  $A$  is a primitive matrix and if  $B \geq A$ , then  $B$  is primitive as well. Lemma 1 is a simple consequence of the following result.

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<sup>4</sup>The authors thank Daniel Liberzon for pointing out a flaw in the original version of this proof, and Sean Meyn for suggesting how to fix it.

**Lemma 2** *Let  $A$  and  $B$  be two  $n \times n$  non-negative matrices. Suppose that the diagonal elements of  $A$  are all positive and let  $\mu_A$  denote the value of the smallest of these. Then*

$$AB \geq \mu_A B \quad BA \geq \mu_A B$$

**Proof:** It is possible to write  $A = \mu_A I + \bar{A}$  where  $\bar{A}$  is non-negative. Then

$$AB = (\mu_A I + \bar{A})B = \mu_A B + \bar{A}B \geq \mu_A B$$

Similarly

$$BA = B(\mu_A I + \bar{A}) = \mu_A B + B\bar{A} \geq \mu_A B$$

This completes the proof. ■

Lemma 2 implies that if  $A$  has a positive diagonal and if  $B$  is primitive, then the products  $AB$  and  $BA$  are both bounded below by a primitive matrix, namely  $\mu_A B$ . Lemma 1 follows at once.

## 2.1 Quadratic Lyapunov Functions

As we've already noted,  $F_p \mathbf{1} = \mathbf{1}$ ,  $p \in \mathcal{P}$ . From this it follows that for any  $n \times (n-1)$  matrix  $P$  with kernel spanned by  $\mathbf{1}$ , the equations

$$PF_p = \bar{F}_p P, \quad p \in \mathcal{P} \tag{12}$$

have unique solutions  $\bar{F}_p$ ,  $p \in \mathcal{P}$ , and moreover that

$$\text{spectrum } F_p = \{1\} \cup \text{spectrum } \bar{F}_p, \quad p \in \mathcal{P} \tag{13}$$

As a consequence of (12) it can easily be seen that for any sequence of indices  $p_0, p_1, \dots, p_i$  in  $\mathcal{P}$ ,

$$\bar{F}_{p_i} \bar{F}_{p_{i-1}} \cdots \bar{F}_{p_0} P = P F_{p_i} F_{p_{i-1}} \cdots F_{p_0} \tag{14}$$

Since  $P$  has full row rank and  $P\mathbf{1} = 0$ , the convergence of a product of the form  $F_{p_i} F_{p_{i-1}} \cdots F_{p_0}$  to  $\mathbf{1}c$  for some row vector  $c$ , is equivalent to convergence of the corresponding product  $\bar{F}_{p_i} \bar{F}_{p_{i-1}} \cdots \bar{F}_{p_0}$  to the zero matrix. Thus, for example, if  $p_0, p_1, \dots$  is an infinite sequence of indices in  $\mathcal{Q}$ , then, in view of Theorem 3,

$$\lim_{i \rightarrow \infty} \bar{F}_{p_i} \bar{F}_{p_{i-1}} \cdots \bar{F}_{p_0} = 0 \tag{15}$$

Some readers might be tempted to think, as we first did, that the validity of (15) could be established directly by showing that the  $\bar{F}_p$  in the product share a common quadratic Lyapunov function. More precisely, (15) would be true if there were a single positive definite matrix  $M$  such that all of the matrices  $\bar{F}_p' M \bar{F}_p - M$ ,  $p \in \mathcal{Q}$  were negative definite. Although each  $\bar{F}_p$ ,  $p \in \mathcal{Q}$  can easily be shown to be discrete-time stable, there are classes of  $F_p$  for which that no such common Lyapunov matrix  $M$  exists. While we've not been able to construct a simple analytical example which demonstrates this, we have been able to determine, for example, that no common quadratic Lyapunov function exists for the class of all  $F_p$  whose associated graphs have 10 vertices and are connected. One can verify that this is so by using semidefinite programming and restricting the check to just those connected graphs on 10 vertices with either 9 or 10 edges.

## 2.2 Generalization

It is possible to interpret the Vicsek model analyzed in the last section as the closed-loop system which results when a suitably defined decentralized feedback law is applied to the  $n$ -agent heading model

$$\theta(t+1) = \theta(t) + u(t) \quad (16)$$

with open-loop control  $u$ . To end up with the Vicsek model,  $u$  would have to be defined as

$$u(t) = -(I + D_{\sigma(t)})^{-1}e(t) \quad (17)$$

where  $e$  is the *average heading error* vector

$$e(t) \triangleq L_{\sigma(t)}\theta(t) \quad (18)$$

and, for each  $p \in \mathcal{P}$ ,  $L_p$  is the symmetric matrix

$$L_p = D_p - A_p \quad (19)$$

known in graph theory as the *Laplacian* of  $\mathbb{G}_p$  [19, 21]. It is easily verified that equations (16) to (19) do indeed define the Vicsek model. We've elected to call  $e$  the average heading error because if  $e(t) = 0$  at some time  $t$ , then the heading of each agent with neighbors at that time will equal the average of the headings of its neighbors.

In the present context, Vicsek's control (17) can be viewed as a special case of a more general decentralized feedback control of the form

$$u(t) = -G_{\sigma(t)}^{-1}L_{\sigma(t)}\theta(t) \quad (20)$$

where for each  $p \in \mathcal{P}$ ,  $G_p$  is a suitably defined, nonsingular diagonal matrix with  $i$ th diagonal element  $g_p^i$ . This, in turn, is an abbreviated description of a system of  $n$  individual agent control laws of the form

$$u_i(t) = -\frac{1}{g_i(t)} \left( n_i(t)\theta_i(t) + \sum_{j \in \mathcal{N}_i(t)} \theta_j(t) \right), \quad i \in \{1, 2, \dots, n\} \quad (21)$$

where for  $i \in \{1, 2, \dots, n\}$ ,  $u_i(t)$  is the  $i$ th entry of  $u(t)$  and  $g_i(t) \triangleq g_{\sigma(t)}^i$ . Application of this control to (16) would result in the closed-loop system

$$\theta(t+1) = \theta(t) - G_{\sigma(t)}^{-1}L_{\sigma(t)}\theta(t) \quad (22)$$

Note that the form of (22) implies that if  $\theta$  and  $\sigma$  were to converge to constant values  $\bar{\theta}$ , and  $\bar{\sigma}$  respectively, then  $\bar{\theta}$  would automatically satisfy  $L_{\bar{\sigma}}\bar{\theta} = 0$ . This means that control (20) automatically forces each agent's heading to converge to the average of its neighbors, if agent headings were to converge at all. In other words, the choice of the  $G_p$  does not effect the requirement that each agent's heading equal the average of the headings of its neighbors, if there is convergence at all. This phenomenon is of course well known in control theory and is a manifestation of the idea of integral control.

The preceding suggests that there might be useful choices for the  $G_p$  alternative to those considered by Vicsek, which also lead to convergence. One such choice turns out to be

$$G_p = gI, \quad p \in \mathcal{P} \quad (23)$$

where  $g$  is any number greater than  $n$ . Our aim is to show that with the  $G_p$  so defined, Theorem 2 continues to be valid. In sharp contrast with the proof technique used in the last section, convergence will be established here using a common quadratic Lyapunov function.

As before, we will use the model

$$\theta(t+1) = F_{\sigma(t)}\theta(t) \quad (24)$$

where, in view of the definition of the  $G_p$  in (23), the  $F_p$  are now symmetric matrices of the form

$$F_p = I - \frac{1}{g}L_p, \quad p \in \mathcal{P} \quad (25)$$

To proceed we need to review a number of well known and easily verified properties of graph Laplacians relevant to the problem at hand. For this, let  $\mathbb{G}$  be any given simple graph with  $n$  vertices. Let  $D$  be a diagonal matrix whose diagonal elements are the valences of  $\mathbb{G}$ 's vertices and write  $A$  for  $\mathbb{G}$ 's adjacency matrix. Then, as noted before, the Laplacian of  $\mathbb{G}$  is the symmetric matrix  $L = D - A$ . The definition of  $L$  clearly implies that  $L\mathbf{1} = 0$ . Thus  $L$  must have an eigenvalue at zero and  $\mathbf{1}$  must be an eigenvector for this eigenvalue. Surprisingly  $L$  is always a positive semidefinite matrix [21]. Thus  $L$  must have a real spectrum consisting of non-negative numbers and at least one of these numbers must be 0. It turns out that the number of connected components of  $\mathbb{G}$  is exactly the same as the multiplicity of  $L$ 's eigenvalue at 0 [21]. Thus  $\mathbb{G}$  is a connected graph just in case  $L$  has exactly one eigenvalue at 0. Note that the trace of  $L$  is the sum of the valences of all vertices of  $\mathbb{G}$ . This number can never exceed  $(n-1)n$  and can attain this high value only for a complete graph. In any event, this property implies that the maximum eigenvalue of  $L$  is never larger than  $n(n-1)$ . Actually the largest eigenvalue of  $L$  can never be larger than  $n$  [21]. This means that the eigenvalues of  $\frac{1}{g}L$  must be smaller than 1 since  $g > n$ . From these properties it clearly follows that the eigenvalues of  $(I - \frac{1}{g}L)$  must all be between 0 and 1, and that if  $\mathbb{G}$  is connected, then all will be strictly less than 1 except for one eigenvalue at 1 with eigenvector  $\mathbf{1}$ . Since each  $F_p$  is of the form  $(I - \frac{1}{g}L)$ , each  $F_p$  possesses all of these properties.

Let  $\sigma$  be a fixed switching signal with value  $p_t \in \mathcal{Q}$  at time  $t \geq 0$ . What we'd like to do is to prove that as  $i \rightarrow \infty$ , the matrix product  $F_{p_i}F_{p_{i-1}} \cdots F_{p_0}$  converges to  $\mathbf{1}c$  for some row vector  $c$ . As noted in the section 2.1, this matrix product will so converge just in case

$$\lim_{i \rightarrow \infty} \bar{F}_{p_i} \bar{F}_{p_{i-1}} \cdots \bar{F}_{p_0} = 0 \quad (26)$$

where as in section 2.1,  $\bar{F}_p$  is the unique solution to  $P F_p = \bar{F}_p P$ ,  $p \in \mathcal{P}$  and  $P$  is any full rank  $(n-1) \times n$  matrix satisfying  $P\mathbf{1} = 0$ . For simplicity and without loss of generality we shall henceforth assume that the rows of  $P$  form a basis for the orthogonal complement of the span of  $\mathbf{e}$ . This means that  $PP' = I$ , that  $\bar{F}_p = P F_p P'$ ,  $p \in \mathcal{P}$ , and thus that each  $\bar{F}_p$  is symmetric. Moreover, in view of (13) and the spectral properties of the  $F_p$ ,  $p \in \mathcal{Q}$ , it is clear that each  $\bar{F}_p$ ,  $p \in \mathcal{Q}$  must have a real spectrum lying strictly inside of the unit circle. This plus symmetry means that for each  $p \in \mathcal{Q}$ ,  $\bar{F}_p - I$  is negative definite, that  $\bar{F}_p' \bar{F}_p - I$  is negative definite and thus that the  $(n-1) \times (n-1)$

identity is a common discrete-time Lyapunov matrix for all such  $\bar{F}_p$ . Using this fact it is straight forward to prove that Theorem 1 holds for system (22) provided the  $G_p$  are defined as in (23) with  $g > n$ .

In general, each  $\bar{F}_p$  is a discrete-time stability matrix only if  $p \in \mathcal{Q}$ . Thus to craft a proof of Theorem 2 for the system described by (22) and (23), one needs to take into account what happens between the times when  $\sigma$  takes values in  $\mathcal{Q}$ . As we've already noted, for all  $p \in \mathcal{P}$ ,  $F_p$ 's spectrum must be real and must be contained in the closed unit circle; because of (13), the same is true of the  $\bar{F}_p$ ,  $p \in \mathcal{P}$ . This plus symmetry means that for each  $p \in \mathcal{P}$ ,  $\bar{F}_p - I$  is negative semi-definite and thus that  $\bar{F}_p' \bar{F}_p - I$  is negative semi-definite. As a consequence,  $(P\theta)' P\theta$  cannot grow on intervals on which  $\sigma$ 's values are not in  $\mathcal{Q}$ . Since  $(P\theta)' P\theta$  must decrease on intervals on which  $\sigma$ 's values are all in  $\mathcal{Q}$ , it is clear that if  $\sigma$  takes values in  $\mathcal{Q}$  infinitely often, then  $P\theta$  must converge to zero.

To summarize, both the Vicsek control defined by  $u = -(I + D_{\sigma(t)})^{-1}e(t)$  and the simplified control given by  $u = -\frac{1}{g}e(t)$  achieve the same emergent behavior. While latter is much easier to analyze than the former and achieves convergence under weaker conditions than the former, it has the disadvantage of not being a true decentralized control because each agent must know an upper bound {i.e.,  $g$ } on the total number of agents within the group. Whether or not this is really a disadvantage, of course depends on what the models are to be used for.

### 3 Leader Following

In this section we consider a modified version of Vicsek's discrete-time system consisting of the same group of  $n$  agents as before, plus one additional agent, labeled 0, which acts as the group's leader. Agent 0 moves at the same constant speed as its  $n$  followers but with a fixed heading  $\theta_0$ . The  $i$ th follower updates its heading just as before, using the average of its own heading plus the headings of its neighbors. The difference now is that each follower's set of neighbors can include the leader and does so whenever the leader is within the follower's neighborhood defining circle of radius  $r$ . Agent  $i$ 's update rule thus is of the form

$$\theta_i(t+1) = \frac{1}{1 + n_i(t) + b_i(t)} \left( \theta_i(t) + \sum_{j \in \mathcal{N}_i(t)} \theta_j(t) + b_i(t)\theta_0 \right) \quad (27)$$

where as before,  $\mathcal{N}_i(t)$  is the set of labels of agent  $i$ 's neighbors from the original group of  $n$  followers, and  $n_i(t)$  is the number of labels within  $\mathcal{N}_i(t)$ . Agent 0's heading is accounted for in the  $i$ th average by defining  $b_i(t)$  to be 1 whenever agent 0 is a neighbor of agent  $i$  and 0 otherwise.

The explicit form of the  $n$  update equations exemplified by (27), depends on the relationships between neighbors which exist at time  $t$ . Like before, each of these relationships can be conveniently described by a simple undirected graph. In this case, each such graph has vertex set  $\{0, 1, 2, \dots, n\}$  and is defined so that  $(i, j)$  is one of the graph's edges just in case agents  $i$  and  $j$  are neighbors. For this purpose we consider an agent - say  $i$  - to be a neighbor of agent 0 whenever agent 0 is a neighbor of agent  $i$ . We will need to consider all possible such graphs. In the sequel we use the symbol  $\bar{\mathcal{P}}$  to denote a set indexing the class of all simple graphs  $\bar{\mathbb{G}}_p$  defined on vertices  $0, 1, 2, \dots, n$ . We will also continue to make reference to the set of all simple graphs on vertices  $1, 2, \dots, n$ . Such

graphs are now viewed as subgraphs of the  $\bar{\mathbb{G}}_p$ . Thus, for  $p \in \bar{\mathcal{P}}$ ,  $\mathbb{G}_p$  now denotes the subgraph obtained from  $\bar{\mathbb{G}}_p$  by deleting vertex 0 and all edges incident on vertex 0.

The set of agent heading update rules defined by (27) can be written in state form. Toward this end, for each  $p \in \bar{\mathcal{P}}$ , let  $A_p$  denote the  $n \times n$  adjacency matrix of the  $n$ -agent graph  $\mathbb{G}_p$  and let  $D_p$  be the corresponding diagonal matrix of valences of  $\mathbb{G}_p$ . Then in matrix terms, (27) becomes

$$\theta(t+1) = (I + D_{\sigma(t)} + B_{\sigma(t)})^{-1}((I + A_{\sigma(t)})\theta(t) + B_{\sigma(t)}\mathbf{1}\theta_0), \quad t \in \{0, 1, 2, \dots\} \quad (28)$$

where  $\sigma : \{0, 1, \dots\} \rightarrow \bar{\mathcal{P}}$  is now a switching signal whose value at time  $t$ , is the index of the graph  $\bar{\mathbb{G}}_p$  representing the agent system's neighbor relationships at time  $t$  and for  $p \in \bar{\mathcal{P}}$ ,  $B_p$  is the  $n \times n$  diagonal matrix whose  $i$ th diagonal element is 1 if  $(i, 0)$  is one of  $\bar{\mathbb{G}}_p$ 's edges and 0 otherwise.

Much like before, our goal here is to show for a large class of switching signals and for any initial set of follower agent headings, that the headings of all  $n$  followers converge to the heading of the leader. As before convergence can only be expected if the leader-follower group is linked together sufficiently often. While convergence does not require the  $\mathbb{G}_p$  encountered along a trajectory to be connected, it does require the corresponding  $\bar{\mathbb{G}}_p$  to be connected, at least some of the time. In the sequel we write  $\bar{\mathcal{Q}}$  for the subset of  $\bar{\mathcal{P}}$  consisting of the indices of those graphs in  $\{\bar{\mathbb{G}}_p : p \in \bar{\mathcal{P}}\}$  which are connected. Our main result on leader following is as follows.

**Theorem 4** *Let  $\theta(0)$  and  $\theta_0$  be fixed and let  $\sigma : \{0, 1, 2, \dots\} \rightarrow \bar{\mathcal{P}}$  be a switching signal for which there exists a positive integer  $T$  large enough so that  $\sigma(t) \in \bar{\mathcal{Q}}$  for at least one value of  $t$  in each time-interval of length  $T$ . Then*

$$\lim_{t \rightarrow \infty} \theta(t) = \theta_0 \mathbf{1} \quad (29)$$

The theorem says that the  $n$  followers in the group eventually follow the groups' leader provided there is a positive integer  $T$  which is large enough so that the leader-follower group is linked together at least once on each time interval of length  $T$ . In the sequel we outline several preliminary ideas upon which the proof of Theorem 4 depends.

To begin, let us note that to prove that (29) holds is equivalent to proving that  $\lim_{t \rightarrow \infty} \epsilon(t) \rightarrow 0$  where  $\epsilon$  is the heading error vector  $\epsilon(t) \triangleq \theta(t) - \theta_0 \mathbf{1}$ . From (28) it is easy to deduce that  $\epsilon$  satisfies the equation

$$\epsilon(t+1) = F_{\sigma(t)}\epsilon(t) \quad (30)$$

where for  $p \in \bar{\mathcal{P}}$ ,  $F_p$  is

$$F_p = (I + D_p + B_p)^{-1}(I + A_p) \quad (31)$$

Note that the partitioned matrices

$$\bar{F}_p \triangleq \begin{bmatrix} F_p & H_p \mathbf{1} \\ 0 & 1 \end{bmatrix}, \quad p \in \bar{\mathcal{P}} \quad (32)$$

are stochastic where, for  $p \in \bar{\mathcal{P}}$ ,

$$H_p \triangleq (I + D_p + B_p)^{-1} B_p \quad (33)$$

To proceed, we need a few more ideas concerned with non-negative matrices. In the sequel we write  $M > N$  whenever  $M - N$  is a positive matrix, where by a *positive matrix* is meant a matrix with all positive entries. For any non-negative matrix  $R$  of any size, we write  $\|R\|$  for the largest of the row sums of  $R$ . Note that  $\|R\|$  is the induced infinity norm of  $R$  and consequently is sub-multiplicative. We denote by  $\lceil R \rceil$ , the matrix obtained by replacing all of  $R$ 's non-zero entries with 1s. Note that  $R > 0$  just in case  $\lceil R \rceil > 0$ . It is also true for any pair of  $n \times n$  non-negative matrices  $A$  and  $B$  with positive diagonal elements, that  $\lceil AB \rceil = \lceil \lceil A \rceil \lceil B \rceil \rceil$ . If, in addition,  $A$  and  $B$  are square and  $A$  has positive diagonal elements, then by Lemma 2,  $\lceil AB \rceil \geq \lceil B \rceil$  and  $\lceil BA \rceil \geq \lceil B \rceil$ .

Let  $p \in \bar{\mathcal{P}}$  be fixed. It is possible to relate the connectedness of  $\bar{\mathbb{G}}_p$  to properties of the matrix pair  $(F_p, H_p \mathbf{1})$ . Let us note first that the indices of the non-zero rows of  $B_p \mathbf{1}$  are precisely the labels of vertices in  $\bar{\mathbb{G}}_p$  which are connected to vertex 0 by paths of length 1. More generally, for any integer  $m > 0$  the indices of the non-zero rows of  $(I + A_p)^{(m-1)} B_p \mathbf{1}$  are the labels of vertices in  $\bar{\mathbb{G}}_p$  connected to vertex 0 by paths of length less than or equal to  $m$ . Since  $\bar{\mathbb{G}}_p$  is a connected graph, it follows that there must be an integer  $m_p > 0$  such that  $(I + A_p)^{(m_p-1)} B_p \mathbf{1} > 0$ . Thus if we define  $\bar{m} = \max\{m_p : p \in \bar{\mathcal{Q}}\}$ , then  $(I + A_p)^{(i-1)} B_p \mathbf{1} > 0$ ,  $i \geq \bar{m}$ ,  $p \in \bar{\mathcal{Q}}$ . Now it is easy to see from the definitions of the  $F_p$  and  $H_p$  in (31) and (33) respectively, that  $\lceil F_p^i H_p \mathbf{1} \rceil = \lceil (I + A_p)^i B_p \mathbf{1} \rceil$ ,  $i \geq 0$ . It thus follows that

$$F_p^{(i-1)} H_p \mathbf{1} > 0, \quad i \geq \bar{m}, \quad p \in \bar{\mathcal{Q}} \quad (34)$$

Now consider the partitioned matrices  $\bar{F}_p$  defined by (32). Since each of these matrices is stochastic and products of stochastic matrices are also stochastic, for each  $p \in \bar{\mathcal{P}}$  and each  $i \geq 1$ ,  $\bar{F}_p^i$  is stochastic. But

$$\bar{F}_p^i = \begin{bmatrix} F_p^i & \sum_{j=1}^i F_p^{(j-1)} H_p \mathbf{1} \\ 0 & 1 \end{bmatrix}, \quad p \in \bar{\mathcal{P}}$$

Moreover, for  $p \in \bar{\mathcal{Q}}$

$$\sum_{j=1}^i F_p^{(j-1)} H_p \mathbf{1} > 0, \quad i \geq \bar{m} \quad (35)$$

because of (34). It follows that for  $p \in \bar{\mathcal{Q}}$ , and any  $i \geq \bar{m}$ , the row sums of  $F_p^i$  must all be less than 1. In other words,

$$\|F_p^i\| < 1, \quad i \geq \bar{m}, \quad p \in \bar{\mathcal{Q}} \quad (36)$$

The following proposition generalizes (36) and is central to the proof of Theorem 4.

**Proposition 1** *Let  $\bar{t} \geq \bar{m}$  be a fixed positive integer. There exists a positive number  $\lambda < 1$ , depending only on  $\bar{t}$  and the  $F_p$ ,  $p \in \bar{\mathcal{P}}$ , for which*

$$\|F_{p_{\bar{t}}} F_{p_{\bar{t}-1}} \cdots F_{p_1}\| < \lambda \quad (37)$$

for every sequence  $p_1, p_2, \dots, p_{\bar{t}}$  which contains a value  $q \in \bar{\mathcal{Q}}$  which occurs in the sequence at least  $\bar{m}$  times.

The proof of this proposition depends on the following basic property of non-negative matrices.

**Lemma 3** *Let  $M_1, M_2, \dots, M_k$  be a finite sequence of  $n \times n$  non-negative matrices whose diagonal entries are all positive. Suppose that  $M$  is a matrix which occurs in the sequence at least  $m > 0$  times. Then*

$$\lceil M_1 M_2 \cdots M_k \rceil \geq \lceil M^m \rceil \quad (38)$$

**Proof:** We claim that for  $j \geq 1$

$$\lceil M_1 M_2 \cdots M_{k_j} \rceil \geq \lceil M^j \rceil \quad (39)$$

provided  $M_1 M_2 \cdots M_{k_j}$  is a product within which  $M$  occurs at least  $j$  times. Suppose  $M_1 M_2 \cdots M_{k_1}$  is a product within which  $M$  occurs at least once. Then  $M_1 M_2 \cdots M_{k_1} = AMB$  where  $A$  and  $B$  are non-negative matrices with positive diagonal elements. By Lemma 2,  $\lceil AMB \rceil \geq \lceil MB \rceil$  and  $\lceil MB \rceil \geq \lceil M \rceil$ . Thus  $\lceil AMB \rceil \geq \lceil M \rceil$  which proves that (39) is true for  $j = 1$ .

Now suppose that (39) holds for  $j \in \{1, 2, \dots, i\}$  and let  $M_1 M_2 \cdots M_{k_{i+1}}$  be a product within which  $M$  occurs at least  $i + 1$  times. We can write  $M_1 M_2 \cdots M_{k_{i+1}} = AMB$  where  $A$  and  $B$  are non-negative matrices with positive diagonal elements and  $A$  is a product within which  $M$  occurs at least  $i$  times. By the inductive hypothesis,  $\lceil A \rceil \geq \lceil M^i \rceil$ . By Lemma 2,  $\lceil AMB \rceil \geq \lceil AM \rceil$ . It follows that  $\lceil AM \rceil = \lceil \lceil A \rceil \lceil M \rceil \rceil \geq \lceil \lceil M^i \rceil \lceil M \rceil \rceil = \lceil M^{i+1} \rceil$  and thus that (39) holds for  $j = i + 1$ . By induction, (39) therefore holds for all  $i \in \{1, 2, \dots, m\}$ . Hence the lemma is true. ■

**Proof of Proposition 1:** It will be enough to prove that

$$\|F_{p_i} F_{p_{i-1}} \cdots F_{p_1}\| < 1 \quad (40)$$

for every sequence  $p_1, p_2, \dots, p_i$  for which there is a value  $q \in \bar{\mathcal{Q}}$  which occurs in the sequence at least  $\bar{m}$  times. For if this is so, then one can define the uniform bound

$$\lambda \triangleq \max_{\mathcal{S}} \|F_{p_{\bar{t}}} F_{p_{\bar{t}-1}} \cdots F_{p_1}\|$$

where  $\mathcal{S}$  is the set of sequences  $p_1, p_2, \dots, p_{\bar{t}}$  of length  $\bar{t}$  with the property that for each such sequence there is a value  $q \in \bar{\mathcal{Q}}$  which occurs in the sequence at least  $\bar{m}$  times. Note that  $\lambda < 1$  if (40) holds, because  $\mathcal{S}$  is a finite set.

Let  $p_1, p_2, \dots, p_i$  be a sequence for which there is a value  $q \in \bar{\mathcal{Q}}$  which occurs in the sequence at least  $\bar{m}$  times. The definition of the  $\bar{F}_p$  in (32) implies that

$$\bar{F}_{p_i} \bar{F}_{p_{i-1}} \cdots \bar{F}_{p_1} = \begin{bmatrix} F_{p_i} F_{p_{i-1}} \cdots F_{p_1} & \sum_{j=1}^i \Phi_{ij} H_{p_j} \mathbf{1} \\ 0 & 1 \end{bmatrix}$$

where  $\Phi_{ii} = I$  and  $\Phi_{ij} = F_{p_i} F_{p_{i-1}} \cdots F_{p_{j+1}}$  for  $j < i$ . Since the  $\bar{F}_p$  are all stochastic,  $\bar{F}_{p_i} \bar{F}_{p_{i-1}} \cdots \bar{F}_{p_1}$  must be stochastic as well. Thus to establish (40) it is sufficient to prove that

$$\sum_{j=1}^i \Phi_{ij} H_{p_j} \mathbf{1} > 0 \quad (41)$$

By assumption, the sequence  $p_1, p_2, \dots, p_i$  has the property that for some  $p \in \bar{\mathcal{Q}}$ , the value  $p$  occurs in the sequence at least  $\bar{m}$  times. Let  $k$  be the smallest integer such that  $p_k = p$ . Since  $p$

occurs at least  $\bar{m}$  times,  $p$  must occur at least  $\bar{m} - 1$  times in the subsequence  $p_{k+1}, p_{k+2}, \dots, p_i$ . It follows from Lemma 3 and the definition of  $\Phi_{ij}$  that  $[\Phi_{ik}] \geq [F_{p_k}^{(\bar{m}-1)}]$ . Thus  $[\Phi_{ik}H_{p_k}\mathbf{1}] \geq [F_{p_k}^{(\bar{m}-1)}H_{p_k}\mathbf{1}]$ . From this and (34) it follows that  $\Phi_{ik}H_{p_k}\mathbf{1} > 0$ . But  $\sum_{j=1}^i \Phi_{ij}H_{p_j}\mathbf{1} \geq \Phi_{ik}H_{p_k}\mathbf{1}$ , so (41) is true. ■

The proof of Proposition 1 actually shows that any finite product  $F_{p_1}F_{p_2}\cdots F_{p_j}$  consisting of at least  $\bar{m}$  matrices, will be a discrete-time stability matrix provided at least one of the  $p_i \in \bar{\mathcal{Q}}$ . From this it is not difficult to see that any finite product  $\bar{F}_{p_1}\bar{F}_{p_2}\cdots\bar{F}_{p_j}$  consisting of at least  $\bar{m}$  matrices will be ergodic provided at least one of the  $p_i \in \bar{\mathcal{Q}}$ . It is possible to use this fact together with Wolfowitz's theorem {Theorem 3} to devise a proof of Theorem 4, much like the proof of Theorem 2 given earlier. On the other hand, it is also possible to give a simple direct proof of Theorem 4, without using Theorem 3, and this is the approach we take.

**Proof of Theorem 4:** The constraints on  $\sigma$  imply that  $\sigma(t)$  takes at least one value in  $\bar{\mathcal{Q}}$  on any interval of length  $T$ . Let  $\bar{n}$  be the number of elements in  $\bar{\mathcal{Q}}$ . Then for any integer  $i > 0$  there must be at least one value  $q \in \bar{\mathcal{Q}}$  which is occurs at least  $i$  times in any sequence of successive values of  $\sigma$  of length  $i\bar{n}T$ . Set  $\bar{T} = \bar{m}\bar{n}T$ . Let  $\Phi(t, s)$  denote the state transition matrix defined by  $\Phi(t, t) = I$ ,  $t \geq 0$  and  $\Phi(t, s) \triangleq F_{\sigma(t-1)}F_{\sigma(t-2)}\cdots F_{\sigma(s)}$ ,  $t > s \geq 0$ . Then  $\epsilon(t) = \Phi(t, 0)\epsilon(0)$ . To complete the theorem's proof, it is therefore enough to show that

$$\lim_{j \rightarrow \infty} \Phi(j\bar{T}, 0) = 0 \quad (42)$$

Clearly  $\Phi(j\bar{T}, 0) = \Phi(j\bar{T}, (j-1)\bar{T}) \cdots \Phi(2\bar{T}, \bar{T})\Phi(\bar{T}, 0)$ . Moreover, for  $i \geq 1$ ,  $[(i-1)\bar{T} + 1, i\bar{T}]$  is an interval on which  $\sigma(t)$  takes at least one value in  $\bar{\mathcal{Q}}$  at least  $\bar{m}$  times. It follows from Proposition 1 and the definition of  $\Phi$  that  $\|\Phi(j\bar{T}, (j-1)\bar{T})\| \leq \lambda < 1$ . Hence  $\|\Phi(j\bar{T}, 0)\| \leq \lambda^j$ ,  $j \geq 1$  from which (42) follows at once. ■

## 4 Leader Following in Continuous Time

Our aim here is to study the convergence properties of the continuous-time version of the leader-follower model discussed in the last section. We begin by noting that the update rule for agent  $i$ 's heading, defined by (27), is what results when the local feedback law

$$u_i(t) = -\frac{1}{1 + n_i(t) + b_i(t)} \left( (n_i(t) + b_i(t))\theta_i(t) - \sum_{j \in \mathcal{N}_i(t)} \theta_j(t) - b_i(t)\theta_0 \right) \quad (43)$$

is applied to the open-loop discrete-time heading model

$$\theta_i(t+1) = \theta_i(t) + u_i(t) \quad (44)$$

The continuous-time analog of (44) is the integrator equation

$$\dot{\theta}_i = u_i \quad (45)$$

where now  $t$  takes values in the real half interval  $[0, \infty)$ . On the other hand, the continuous time analog of (43) has exactly the same form as (43), except in the continuous time case,  $n_i(t), b_i(t)$ ,

and  $\theta_i(t)$  are continuous-time variables. Unfortunately, in continuous time control laws of this form can lead to chattering because neighbor relations can change abruptly with changes in agents' positions. One way to avoid this problem is to introduce dwell time, much as was done in [22]. What this means in the present context is that each agent is constrained to change its control law only at discrete times. In particular, instead of using (43), to avoid chatter agent  $i$  would use a hybrid control law of the form

$$u_i(t) = -\frac{1}{1 + n_i(t_{ik}) + b_i(t_{ik})} \left( (n_i(t_{ik}) + b_i(t_{ik}))\theta_i(t) - \sum_{j \in \mathcal{N}_i(t_{ik})} \theta_j(t) - b_i(t_{ik})\theta_0 \right), \quad t \in [t_{ik}, t_{ik} + \tau_i) \quad (46)$$

where  $\tau_i$  is a pre-specified positive number called a *dwell time* and  $t_0, t_1, \dots$  is an infinite time sequence such that  $t_{i(k+1)} - t_{ik} = \tau_i$ ,  $k \geq 0$ . In the sequel we will analyze controls of this form subject to two simplifying assumptions. First we will assume that all  $n$  agents use the same dwell time which we henceforth denote by  $\tau_D$ . Second we assume the agents are synchronized in the sense that  $t_{ik} = t_{jk}$  for all  $i, j \in \{1, 2, \dots, n\}$  and all  $k \geq 0$ . These assumptions enable us to write  $u$  as

$$u = -(I + D_\sigma + B_\sigma)^{-1}((L_\sigma + B_\sigma)\theta - B_\sigma \mathbf{1}\theta_0) \quad (47)$$

where  $\bar{\mathcal{P}}, D_p, B_p$  and  $A_p$  are as before,  $L_p = D_p - A_p$  is the Laplacian of  $\mathbb{G}_p$ , and  $\sigma : [0, \infty) \rightarrow \bar{\mathcal{P}}$  is a piecewise constant switching signal with successive switching times separated by  $\tau_D$  time units. Application of this control to the vector version of (45) results in the closed-loop continuous-time leader-follower model

$$\dot{\theta} = -(I + D_\sigma + B_\sigma)^{-1}((L_\sigma + B_\sigma)\theta - B_\sigma \mathbf{1}\theta_0) \quad (48)$$

Much like before, our goal here is to show for a large class of switching signals and for any initial set of follower agent headings, that the headings of all  $n$  followers converge to the heading of the leader. However unlike the discrete-time case which requires there to be an integer  $T$  large enough so that  $\sigma(t) \in \bar{\mathcal{Q}}$  at least once on each interval of length  $T$ , we shall require only that  $\sigma$ 's sequence of switching times  $t_1, t_2, \dots$  contain an infinite subsequence  $t_{i_1}, t_{i_2}, \dots$  on which  $\sigma$  takes all of its values in  $\bar{\mathcal{Q}}$ .

**Theorem 5** *Let  $\tau_D > 0$ ,  $\theta(0)$  and  $\theta_0$  be fixed and let  $\sigma : [0, \infty) \rightarrow \bar{\mathcal{P}}$  be a piecewise-constant switching signal whose switching times  $t_1, t_2, \dots$  satisfy  $t_{i+1} - t_i \geq \tau_D$ ,  $i \geq 1$ . If there is an infinite subsequence  $t_{i_1}, t_{i_2}, \dots$  such that  $\sigma(t_{i_j}) \in \bar{\mathcal{Q}}$ ,  $j \geq 1$ , then*

$$\lim_{t \rightarrow \infty} \theta(t) = \theta_0 \mathbf{1} \quad (49)$$

Theorem 5 states that  $\theta$  will converge to  $\theta_0 \mathbf{1}$ , no matter what the value of  $\tau_D$ , so long as  $\tau_D$  is greater than zero. This is in sharp contrast to other convergence results involving dwell time switching such as those given in [23], which hold only for sufficiently large values of  $\tau_D$ . Theorem 5 is a more or less obvious consequence of the following lemma.

**Lemma 4** For each  $p \in \bar{\mathcal{P}}$  and each finite  $t > 0$ ,

$$\|e^{-(I+D_p+B_p)^{-1}(L_p+B_p)t}\| \leq 1 \quad (50)$$

If, in addition,  $p \in \bar{\mathcal{Q}}$  then (50) holds with a strict inequality.

**Proof of Theorem 5:** For  $i \geq 1$ , set  $M_i = -(I + D_{\sigma(t_i)} + B_{\sigma(t_i)})^{-1}(L_{\sigma(t_i)} + B_{\sigma(t_i)})(t_{i+1} - t_i)$ . From Lemma 4, the definition of the  $t_{i_j}$  and the assumption that  $t_{i+1} - t_i \geq \tau_D$ ,  $i \geq 1$ , it follows that

$$\|e^{M_i}\| \leq 1, \quad i \geq 1 \quad \text{and} \quad \|e^{M_{i_j}}\| \leq \lambda, \quad j \geq 1 \quad (51)$$

where

$$\lambda = \max_{p \in \bar{\mathcal{Q}}} \|e^{-(I+D_p+B_p)^{-1}(L_p+B_p)\tau_D}\|$$

Note that  $\lambda < 1$  because for  $q \in \bar{\mathcal{Q}}$ , the inequality in (50) is strict and because  $\bar{\mathcal{Q}}$  is a finite set.

Set  $\bar{\theta}(t) = \theta(t) - \mathbf{1}\theta_0$  and note that

$$\dot{\bar{\theta}} = -(I + D_{\sigma} + B_{\sigma})^{-1}(L_{\sigma} + B_{\sigma})\bar{\theta}$$

because of (48). Let  $\Phi(t, \mu)$  be the state transition matrix of  $-(I + D_{\sigma(t)} + B_{\sigma(t)})^{-1}(L_{\sigma(t)} + B_{\sigma(t)})$ . Then  $\bar{\theta}(t) = \Phi(t, 0)\bar{\theta}(0)$ . To complete the proof it is therefore enough to show that

$$\|\Phi(t_{i_j}, t_{i_1})\| \leq \lambda^{j-1}, \quad j \geq 1 \quad (52)$$

In view of the definitions of the  $M_{i_j}$ ,

$$\Phi(t_{i_{j+1}}, t_{i_j}) = \prod_{k=i_j}^{i_{j+1}-1} e^{M_k}, \quad j \geq 1$$

Therefore

$$\|\Phi(t_{i_{j+1}}, t_{i_j})\| = \left\| \prod_{k=i_j}^{i_{j+1}-1} e^{M_k} \right\| \leq \prod_{k=i_j}^{i_{j+1}-1} \|e^{M_k}\|, \quad j \geq 1$$

Thus

$$\|\Phi(t_{i_{j+1}}, t_{i_j})\| \leq \|e^{M_{i_j}}\| \leq \lambda, \quad j \geq 1 \quad (53)$$

because of (51). But

$$\Phi(t_{i_j}, t_1) = \Phi(t_{i_j}, t_{i_{j-1}}) \cdots \Phi(t_{i_2}, t_{i_1}) e^{M_{i_1-1}} \cdots e^{M_1}$$

so

$$\|\Phi(t_{i_j}, t_1)\| \leq \|\Phi(t_{i_j}, t_{i_{j-1}})\| \cdots \|\Phi(t_{i_2}, t_{i_1})\| \|e^{M_{i_1-1}}\| \cdots \|e^{M_1}\|$$

Therefore

$$\|\Phi(t_{i_j}, t_1)\| \leq \|\Phi(t_{i_j}, t_{i_{j-1}})\| \cdots \|\Phi(t_{i_2}, t_{i_1})\|$$

because of (51). From this and (53), it now follows that (52) is true. ■

**Proof of Lemma 4:** Fix  $t > 0$  and  $p \in \bar{\mathcal{P}}$ . Observe first that

$$-(I + D_p + B_p)^{-1}(L_p + B_p) = F_p - I \quad (54)$$

where  $F_p$  is the matrix  $F_p = (I + D_p + B_p)^{-1}(I + A_p)$ . As noted previously, the partitioned matrix

$$\bar{F}_p \triangleq \begin{bmatrix} F_p & H_p \mathbf{1} \\ 0 & 1 \end{bmatrix} \quad (55)$$

originally defined in (32), is stochastic with positive diagonal elements as are the matrices

$$\bar{F}_p^i = \begin{bmatrix} F_p^i & \sum_{j=1}^i F_p^{(j-1)} H_p \mathbf{1} \\ 0 & 1 \end{bmatrix}, \quad i \geq 1 \quad (56)$$

Since

$$e^{\bar{F}_p t} = \sum_{i=0}^{\infty} \frac{(t \bar{F}_p)^i}{i!} \quad (57)$$

$e^{\bar{F}_p t}$  must also be nonnegative with positive diagonal elements. But  $e^{(\bar{F}_p - \bar{I})t} = e^{-t} e^{\bar{F}_p t}$ , so the same must be true of  $e^{(\bar{F}_p - \bar{I})t}$ . Moreover  $(\bar{F}_p - \bar{I})\mathbf{1} = 0$  which means that  $e^{(\bar{F}_p - \bar{I})t}\mathbf{1} = e^0\mathbf{1} = \mathbf{1}$  and thus that  $e^{(\bar{F}_p - \bar{I})t}$  is row stochastic. In summary,  $e^{(\bar{F}_p - \bar{I})t}$  is a row stochastic matrix with positive diagonal entries.

Equations (55) - (57) imply that

$$e^{\bar{F}_p t} = \begin{bmatrix} e^{F_p t} & k_p \\ 0 & e^t \end{bmatrix}$$

where

$$k_p = \sum_{i=0}^{\infty} \frac{t^i}{i!} \sum_{j=1}^i F_p^{(j-1)} H_p \mathbf{1} \quad (58)$$

Therefore

$$e^{(\bar{F}_p - \bar{I})t} = \begin{bmatrix} e^{(F_p - I)t} & k_p \\ 0 & 1 \end{bmatrix} \quad (59)$$

But  $e^{(\bar{F}_p - \bar{I})t}$  is row-stochastic, so  $e^{(F_p - I)t}$  must have its row sums all bounded above by 1. From this and (54) it thus follows that (50) is true.

Now suppose that  $p \in \bar{\mathcal{Q}}$ . Then, as noted previously in (35),

$$\sum_{j=1}^i F_p^{(j-1)} H_p \mathbf{1} > 0, \quad i \geq \bar{m}$$

Therefore, in view of (58),  $k_p > 0$ . From this, (59), and the fact that  $e^{\bar{F}_p t}$  is row-stochastic, it follows that then the row sums of  $e^{(F_p - I)t}$  must all be strictly less than 1. Hence  $\|e^{(F_p - I)t}\| < 1$ . This and (54) therefore imply that (50) holds with a strict inequality. ■

## 5 Concluding Remarks

The convergence proof for Vicsek's model presented in Section 2 relies heavily on Wolfowitz's theorem. By generalizing some of the constructions Wolfowitz used in his proof, it is possible to develop a convergence result for a continuous-time analog of the Vicsek model which is quite similar to Theorem 5.

The convergence results we've derived for both the Vicsek model and for the discrete-time leader follower model, are for classes of switching signals which are probably more restrictive than they need to be. In particular, in both cases it may be enough to assume only that the switching signals in question revisit indices of connected graphs infinitely often.

In studying continuous-time leader-following, we imposed the requirement that all followers use the same dwell time. This is not really necessary. In particular, without much additional effort it can be shown that Theorem 5 remains true under the relatively mild assumption that all agents use dwell times which are rationally related. In contrast, the synchronization assumption may be more difficult to relax. Although convergence is still likely without synchronization, the aperiodic nature of  $\sigma$ 's switching times which could result, make the analysis problem more challenging.

The models we have analyzed are of course very simple and as a consequence, they are probably not really descriptive of actual bird-flocking, fish schooling, or even the coordinated movements of envisioned groups of mobile robots. Nonetheless, these models do seem to exhibit some of the rudimentary behaviors of large groups of mobile autonomous agents and for this reason they serve as a natural starting point for the analytical study of more realistic models. It is clear from the developments in this paper, that ideas from graph theory and dynamical system theory will play a central role in both the analysis of such biologically inspired models and in the synthesis of provably correct distributed control laws which produce such emergent behaviors in man-made systems.

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