Data Rate Theorem for Stabilization over Time-Varying Feedback Channels

Workshop on Frontiers in Distributed

Communication, Sensing and Control

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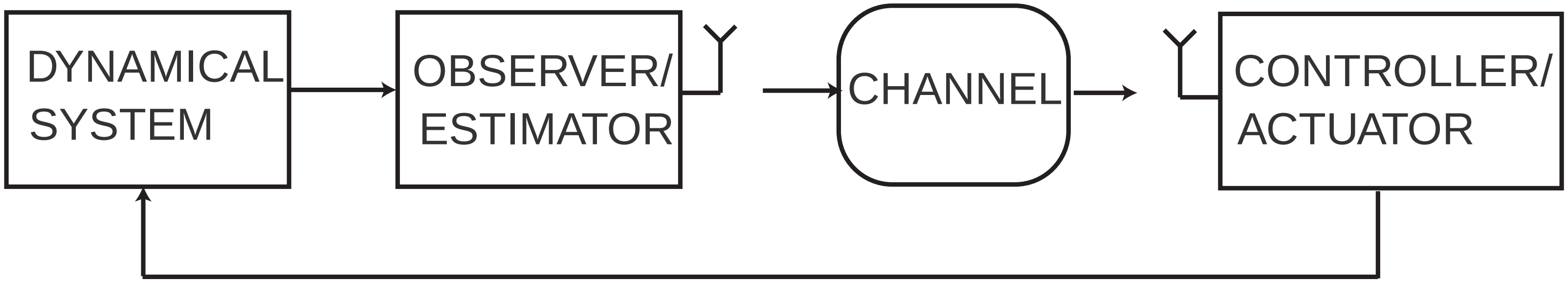
(joint work with P. Minero, S. Dey and G. Nair)

Motivation

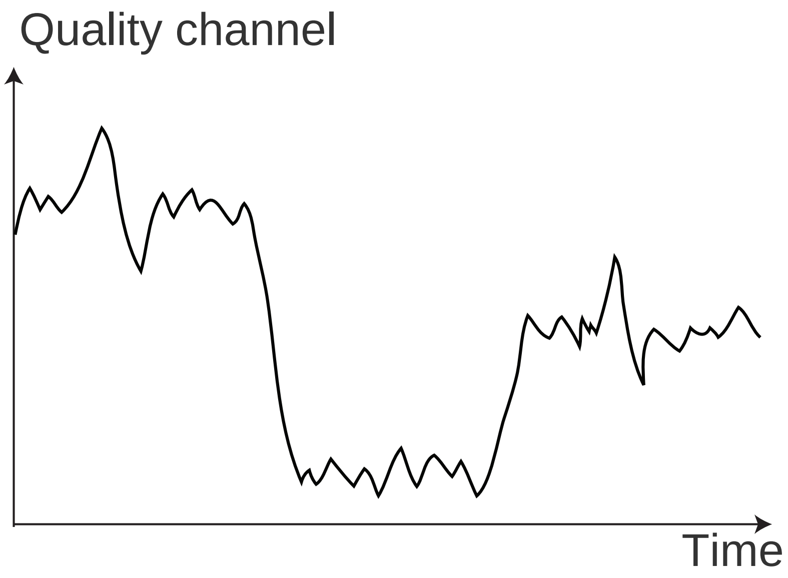
Abstraction


DYNAMICAL SYSTEM

OBSERVER/ ESTIMATOR

CHANNEL

CONTROLLER/ ACTUATOR



Packet loss point of view

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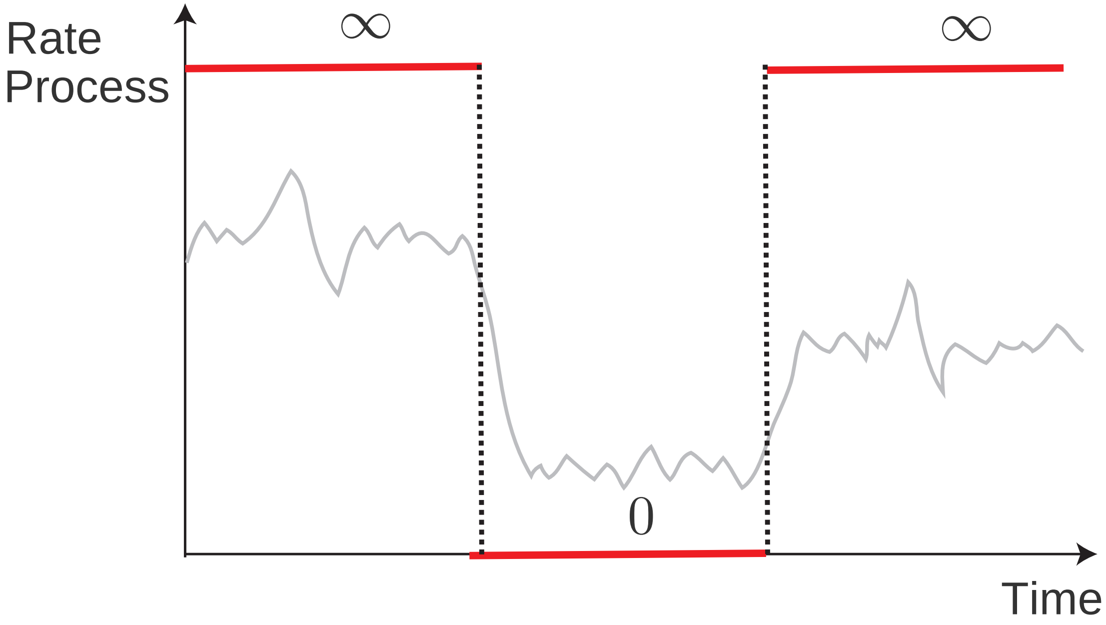
- Rate process:

$$R_k = \begin{cases} \infty & \text{w.p. } 1-p \\ 0 & \text{w.p. } p \end{cases}$$



- There is a critical dropout probability p for estimation (Sinopoli *et al* 2004) and control (Schenato *et al* 2007, Gupta *et al* 2007):

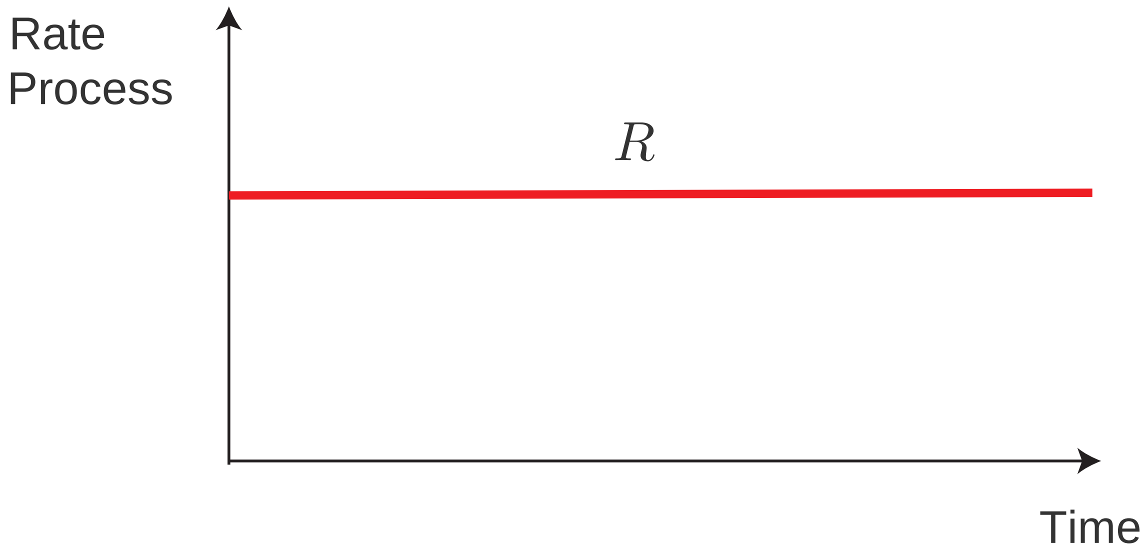
$$p < \frac{1}{\max_i |\lambda_i|^2}.$$

Rate limited feedback point of view

Rate limited feedback point of view

- Rate process:

$$R_k = R$$



- Data rate theorem (Tatikonda-Mitter 2002):

$$R > \sum_u m_u \log_2 |\lambda_u|$$

Can we merge the two approaches?

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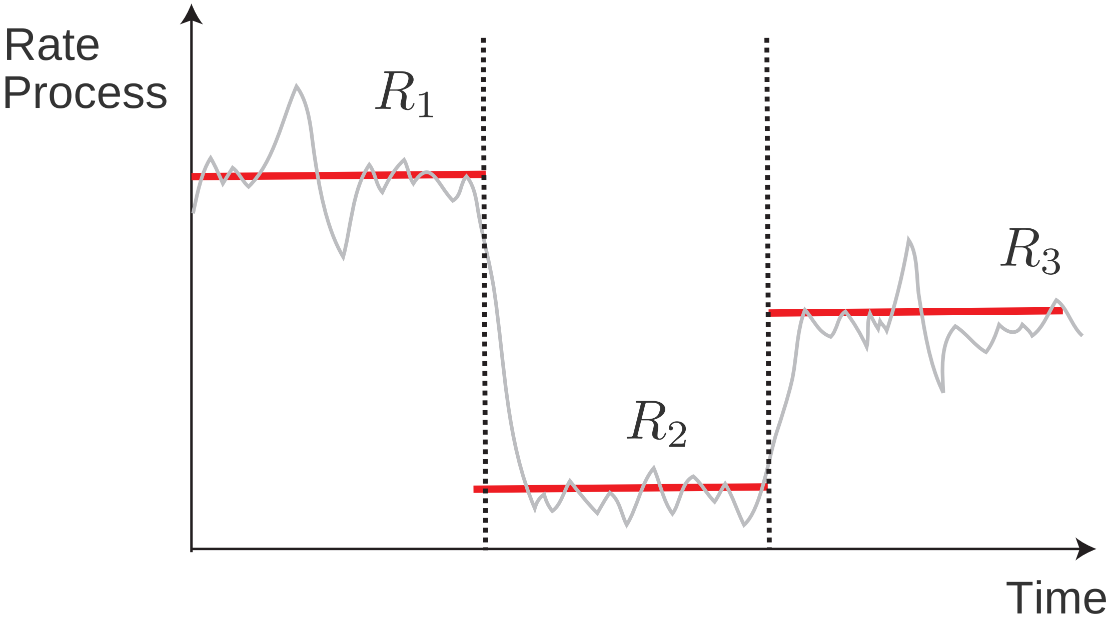
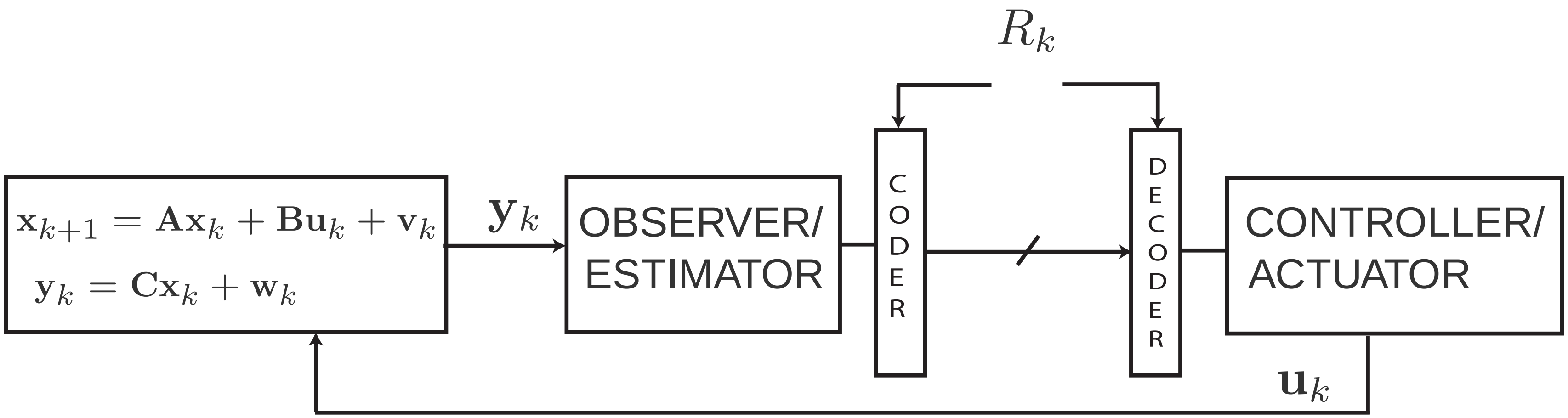
- Stochastic time-varying rate limited feedback: rate process $\{R_k\}$ i.i.d. according to a random variable R.



- **Special cases**: R is deterministic, R is 0 or ∞

Problem formulation



- A set of unstable eigenmodes $|\lambda_1| \geq 1, \cdots, |\lambda_n| \geq 1$.
- Problem: conditions on R to achieve second moment stability:

$$\sup_k \mathbb{E}\left[\| \mathbf{x}_k \|^2\right] < \infty.$$

Problem formulation

- Distributions of $\mathbf{x}_0$ and system disturbances ($\mathbf{v}$ and $\mathbf{w}$):
  - Unbounded support
  - A bounded moment > 2 (e.g. Gaussian distribution)

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  - The case of $R_k = 0$ can be thought as a packet loss *with* acknowledgment at the transmitter.

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- Coder has access to more information than the decoder (Classical information structure).

Related works

	Decoding errors	**Disturbances**	**Time-varying rate**
Tatikonda-Mitter (*Tran AC 02*)	No	Bounded	No
Nair-Evans (*SIAM J.C. 04*)	No	Unbounded	No
Yuksel-Basar (*CDC 05*)	Yes	Unbounded	No
Martins-Dahleh-Elia (*Tran AC 06*)	No	Bounded	Yes
Sahai-Mitter (*Tran IT 06*)	Yes	Bounded	Yes
This work	No	Unbounded	Yes

Summary of results

- **Scalar case**:
  - Necessary and sufficient condition for second moment stabilizability.

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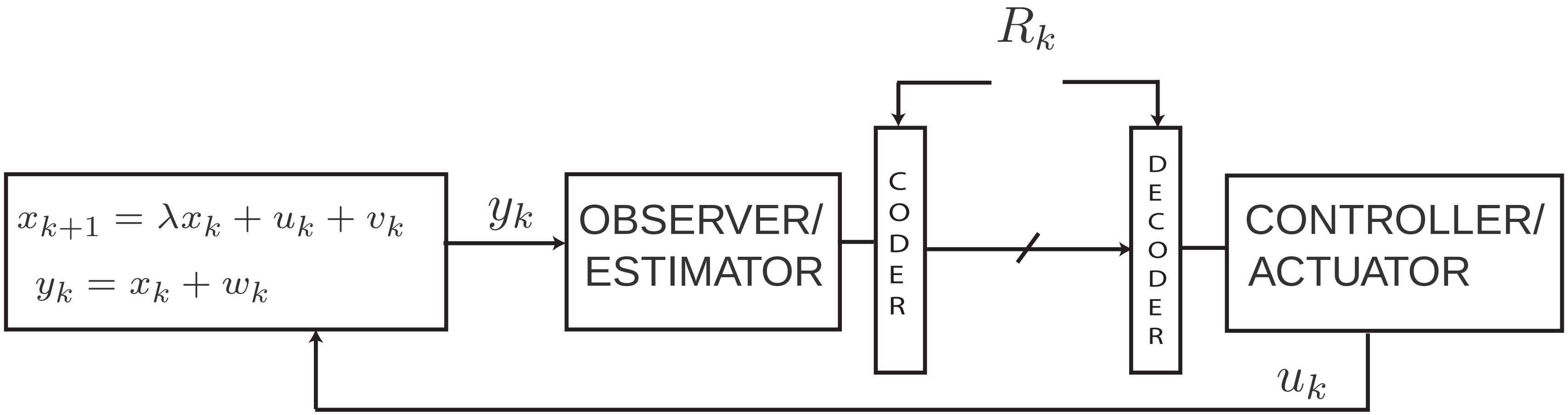
- **Scalar case**:
  - Necessary and sufficient condition for second moment stabilizability.
- **Vector case**: we ‘sandwich’ the stabilizability region between two polytopes.
  - Necessity:
    - The outer polytope has a special geometric structure.
  - Sufficiency:
    - The inner and outer polytopes coincide in some special cases.

Scalar case



R_k

$x_{k+1} = \lambda x_k + u_k + v_k$

$y_k = x_k + w_k$

y_k

OBSERVER/ ESTIMATOR

CODER

DECODER

CONTROLLER/ ACTUATOR

u_k

- Unstable system, so $|\lambda| \geq 1$.

Theorem 1. *Necessary and sufficient condition for stabilization in the mean square sense is that*

$$\mathbb{E}\left[\frac{|\lambda|^2}{2^{2R}}\right] < 1.$$

Scalar case: $\mathbb{E}\left[\frac{|\lambda|^2}{2^{2R}}\right] < 1$

- At each time step, uncertainty volume

$$\uparrow\ |\lambda|^2, \quad \downarrow \frac{1}{2^{2R}}$$

If the ratio, on *average*, is ≥ 1, then the information sent cannot compensate the system dynamics.

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- **Deterministic rate**: we recover the data rate theorem

$$R > \log_2 |\lambda|.$$

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- **Deterministic rate**: we recover the data rate theorem

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- **Packet loss**: we recover the critical dropout probability

$$\mathbb{E}\left[\frac{|\lambda|^2}{2^{2R}}\right] = p\frac{|\lambda|^2}{2^0} + (1-p)\frac{|\lambda|^2}{2^{2r}} < 1 \Rightarrow p < \frac{1}{|\lambda|^2}, \quad \text{as } r \to \infty$$

Scalar case: proofs

- **Necessity:** Using the entropy power inequality we establish the following recursion

$$\mathbb{E}[x_k^2] \geq \mathbb{E}\left[\frac{|\lambda|^2}{2^{2R}}\right] \mathbb{E}[x_{k-1}^2] + \text{const.},$$

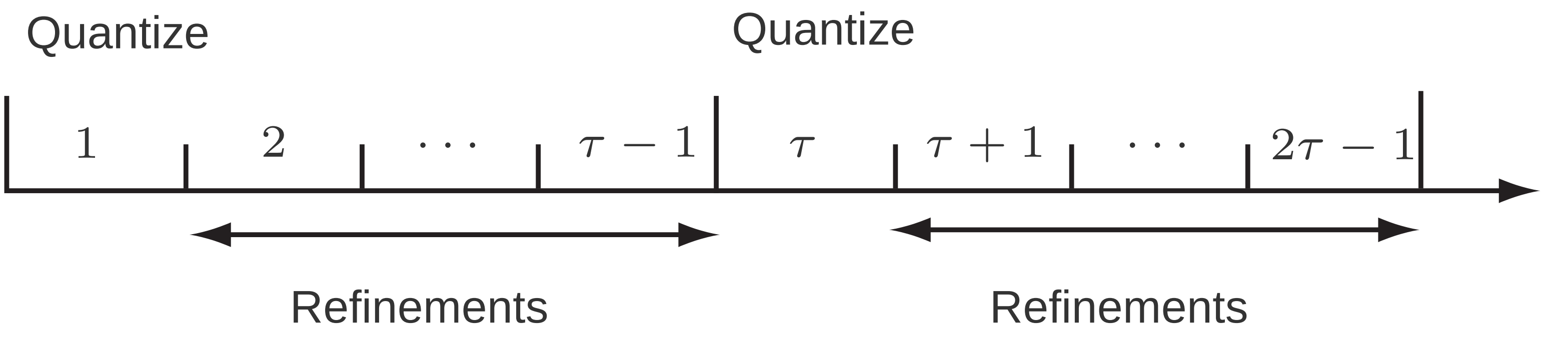
Thus,

$$\sup_k \mathbb{E}[x_k^2] < \infty \Rightarrow \mathbb{E}\left[\frac{|\lambda|^2}{2^{2R}}\right] < 1.$$

- **Sufficiency:** Design an adaptive quantizer, avoid saturation, high resolution through successive refinements.

Successive refinements

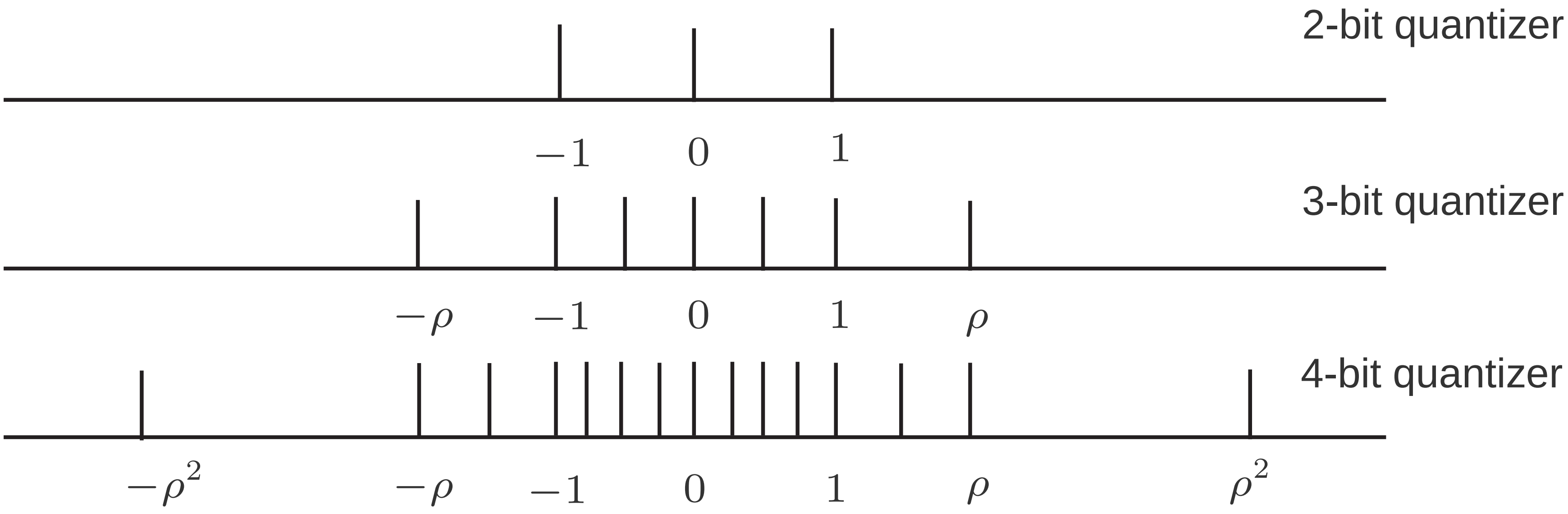
- Divide time into cycles of *fixed* duration τ (of our choice)
- Observe the system at the beginning of each cycle and send an initial estimate
- During the remaining of a cycle, ‘refine’ the initial estimate.



- Number of bits per cycle is a random variable, dependent on the rate process

Successive refinements

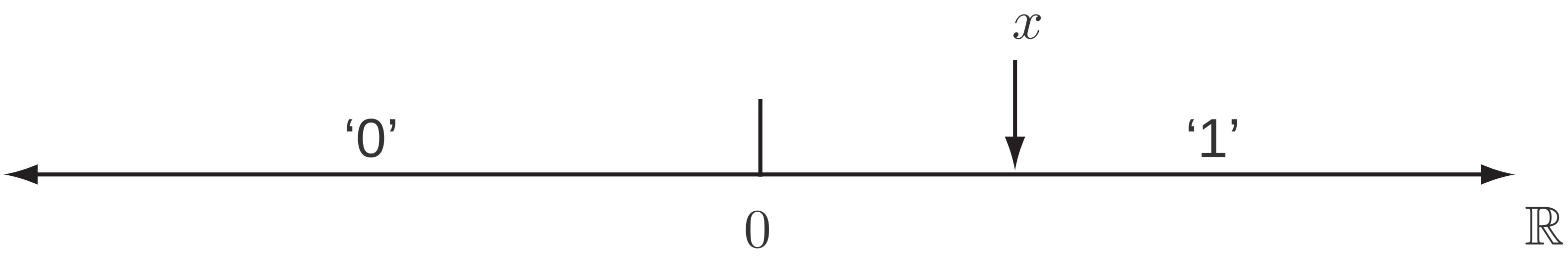
- Quantizer can be generated *recursively* from a 2-bit quantizer, as follows



- ρ is a parameter chosen dependent on the distribution of the disturbances.

Successive refinements: example

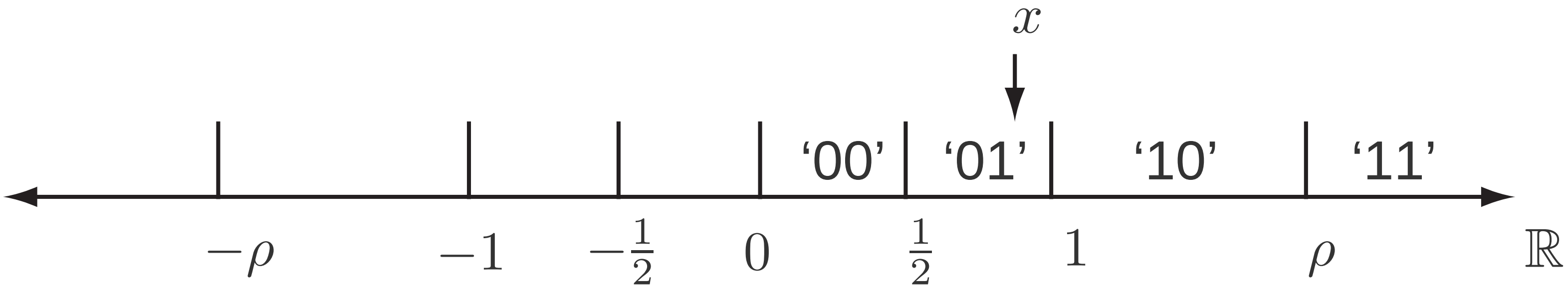
- Suppose we need to quantize a positive real value x.
- At time $k = 1$, suppose $R_1 = 1$.



- With one bit of information, the decoder knows that $x > 0$

Successive refinements: example

- At time $k = 2$, suppose $R_2 = 2$: coder and decoder partition the real axis according to a 3-bit quantizer.

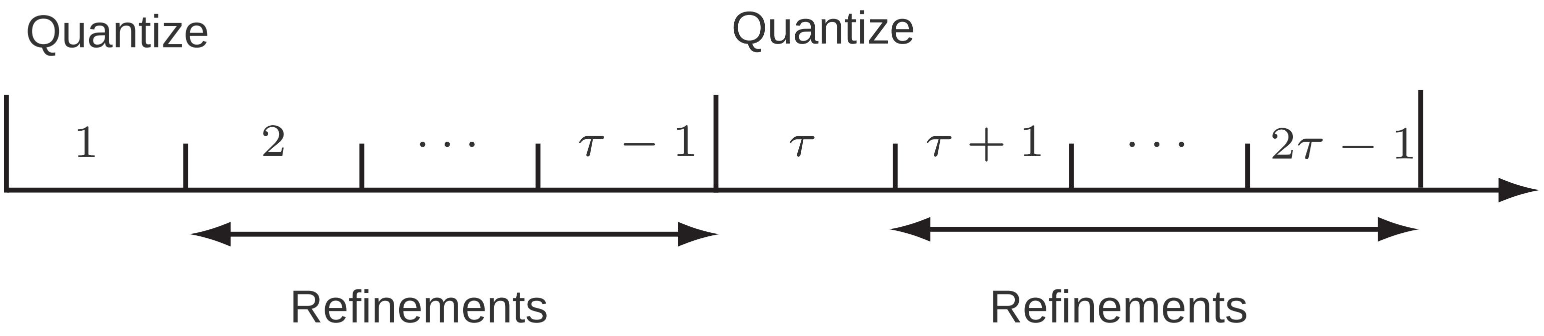


- However, coder and decoded only 'label' the partitions in the positive real line (2 bits suffice).

- After receiving '01', the decoder knows that $x \in \left[\frac{1}{2}, 1\right)$. The initial estimate $x \in [0, \infty)$ has been refined.

- The scheme works as if we had known ahead of time that $R_1 + R_2 = 3$.

Sufficiency: analysis



Quantize

Quantize

1 | 2 | $\cdots$ | $\tau - 1$ | τ | $\tau + 1$ | $\cdots$ | $2\tau - 1$

Refinements

Refinements

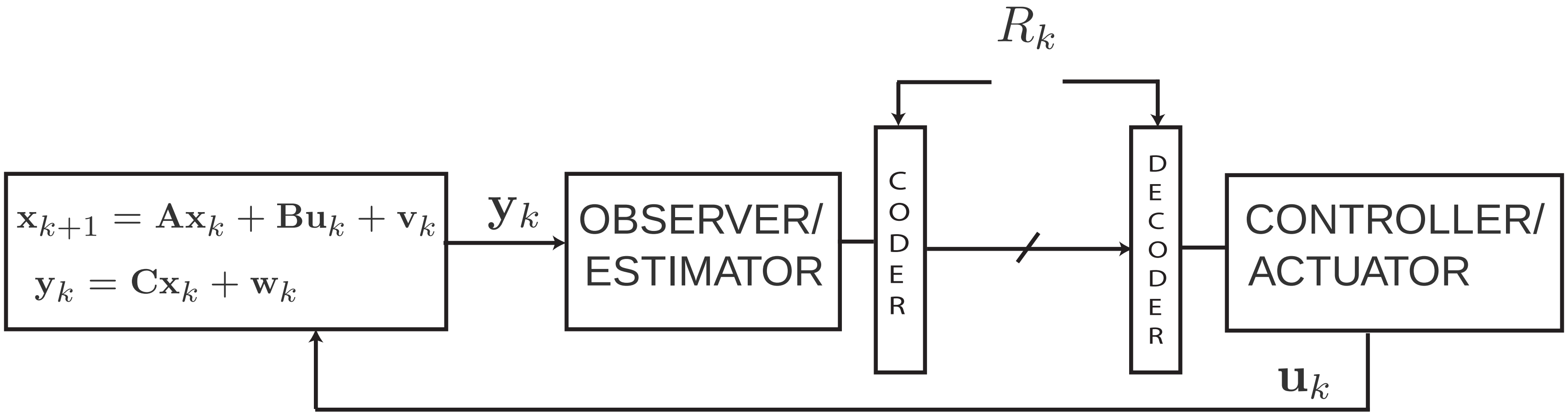
- We find a recursion for $\mathbb{E}[x_{k\tau}^2]$

$$\mathbb{E}[x_{k\tau}^2] \leq \text{const} \left(\mathbb{E}\left[\frac{|\lambda|^2}{2^{2R}} \right] \right)^{\tau} \mathbb{E}[x_{(k-1)\tau}^2] + \text{const.}$$

- Thus, for τ large enough

$$\mathbb{E}\left[\frac{|\lambda|^2}{2^{2R}} \right] < 1 \Rightarrow \text{const} \left(\mathbb{E}\left[\frac{|\lambda|^2}{2^{2R}} \right] \right)^{\tau} < 1.$$

Vector case



- n unstable sub-mode associated to eigenvalues $|\lambda_1| \geq 1, |\lambda_2| \geq 1, \cdots, |\lambda_n| \geq 1$.
- Sub-mode i has multiplicity m_i.
- $(\mathbf{A}, \mathbf{B})$ is reachable and $(\mathbf{C}, \mathbf{A})$ observable.

Vector case: necessity

Theorem 2. *Necessary condition for stabilizability in the mean square sense is that* $(\log_2|\lambda_1|, \ldots, \log_2|\lambda_n|) \in \mathbb{R}^n_+$ *are inside the region determined by the following set of inequalities*

$$\sum_{i=1}^{n} s_i \log_2|\lambda_i| < -\frac{\sum_i s_i}{2} \log_2 \mathbb{E}\left[2^{-\frac{2}{\sum_i s_i}R}\right],$$

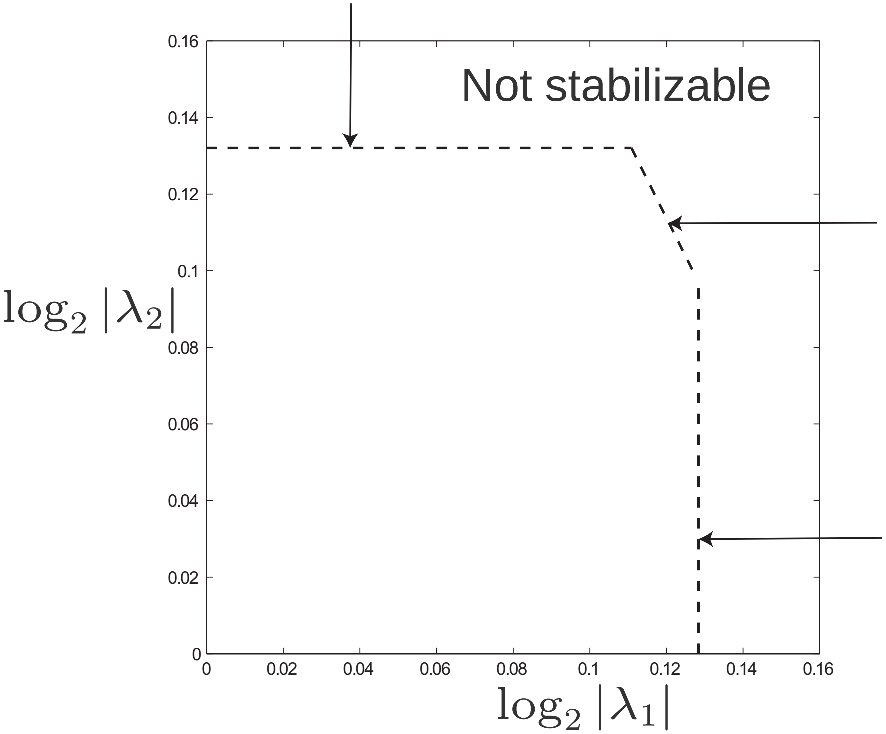
where $\mathbf{s} = [s_1, \cdots, s_n]^T \neq \mathbf{0}$, $s_i \in \{0, \ldots, m_i\}$, $i = 1, \cdots, n$.

Vector case: necessity

Example 1. *Suppose*

$$A = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix},$$

so $m_1 = 2$ *and* $m_2 = 1$*.*



$$\log |\lambda_2| < -\tfrac{1}{2} \log_2 \mathbb{E}\left[2^{-2R}\right]$$

$$2 \log |\lambda_1| + \log |\lambda_2| < -\tfrac{3}{2} \log_2 \mathbb{E}\left[2^{-\frac{2}{3}R}\right]$$

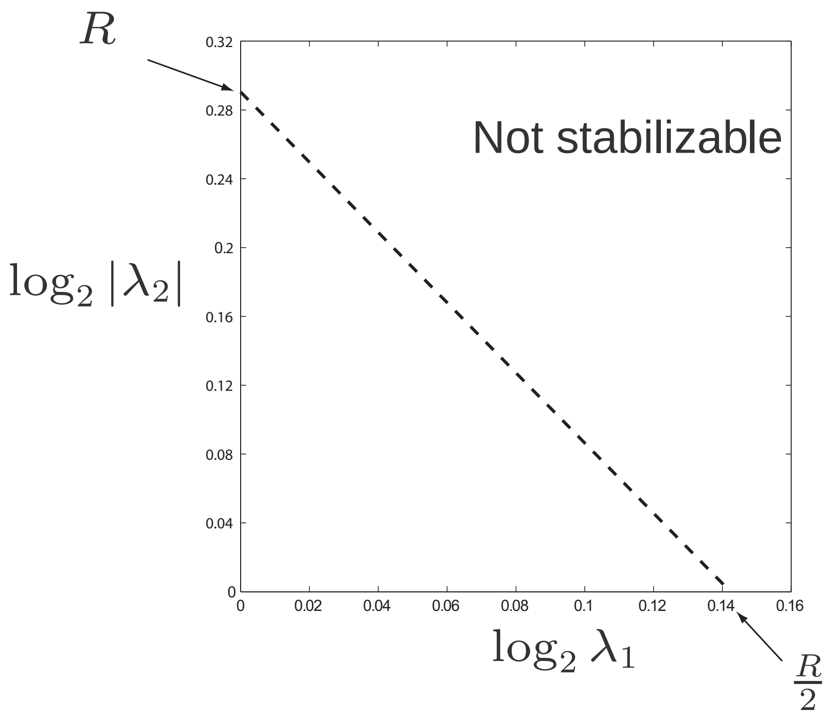
$$2 \log |\lambda_1| < -\log_2 \mathbb{E}\left[2^{-R}\right]$$

Special case: deterministic rate

- We recover the necessity of the data rate theorem

$$2\log|\lambda_1| + \log|\lambda_2| < R.$$

- Polyhedron reduces to a hyperplane

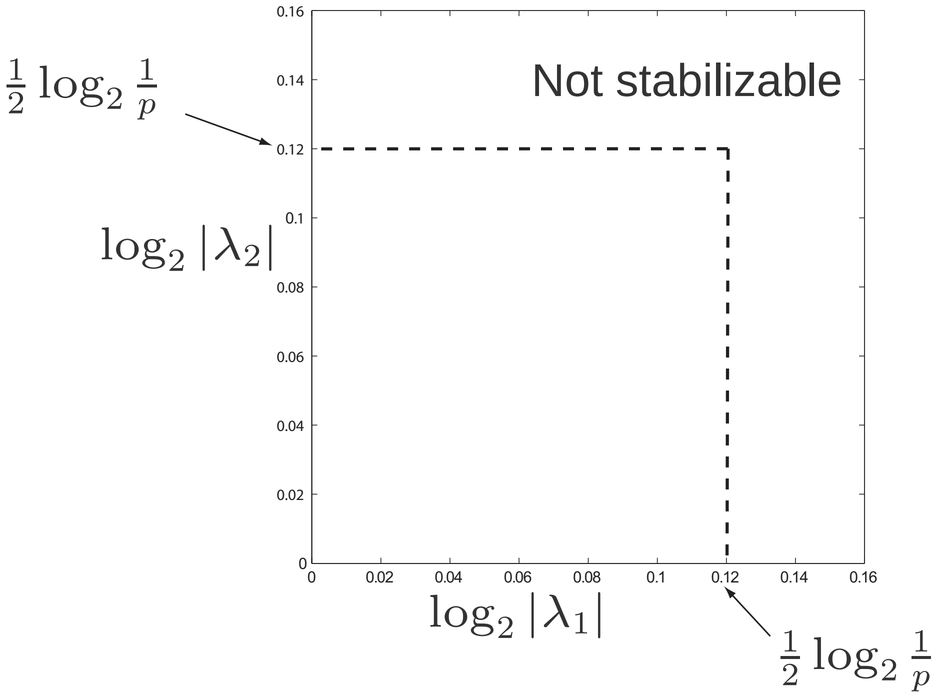


Special case: packet erasure

- We recover the necessity on the critical dropout probability

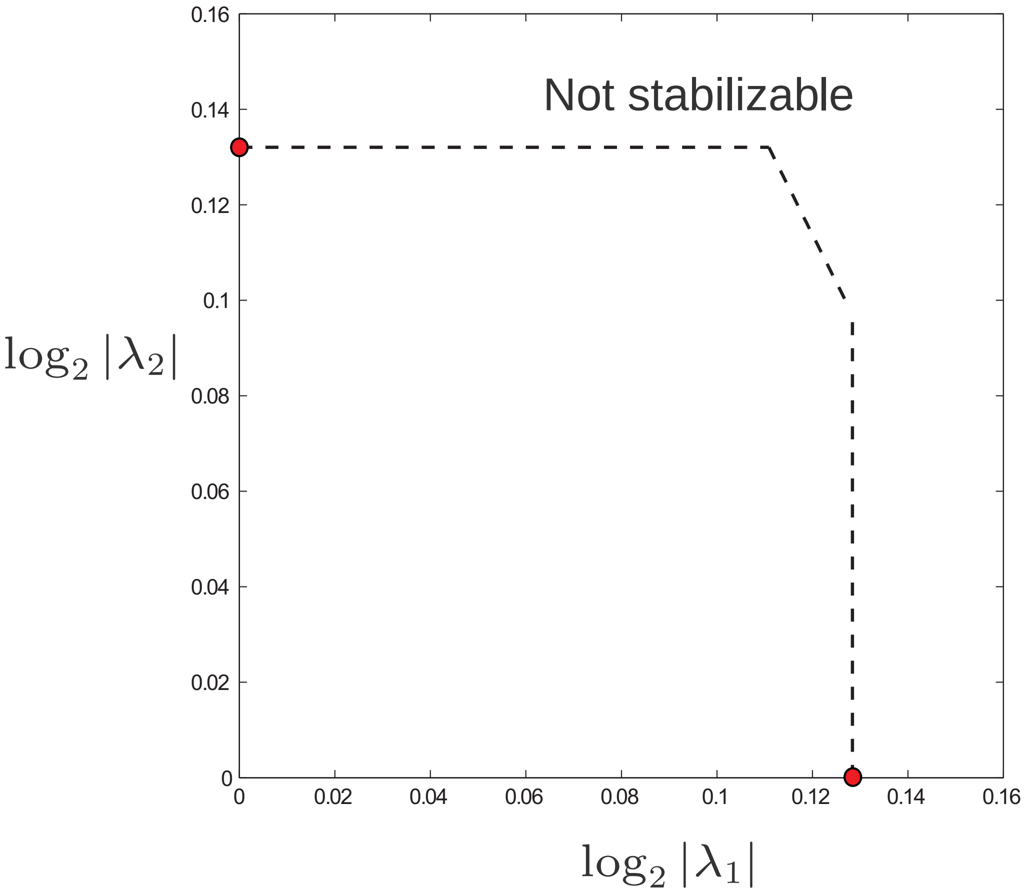
$$p < \frac{1}{\max_i |\lambda_i|^2}.$$

- Polyhedron reduces to a hypercube



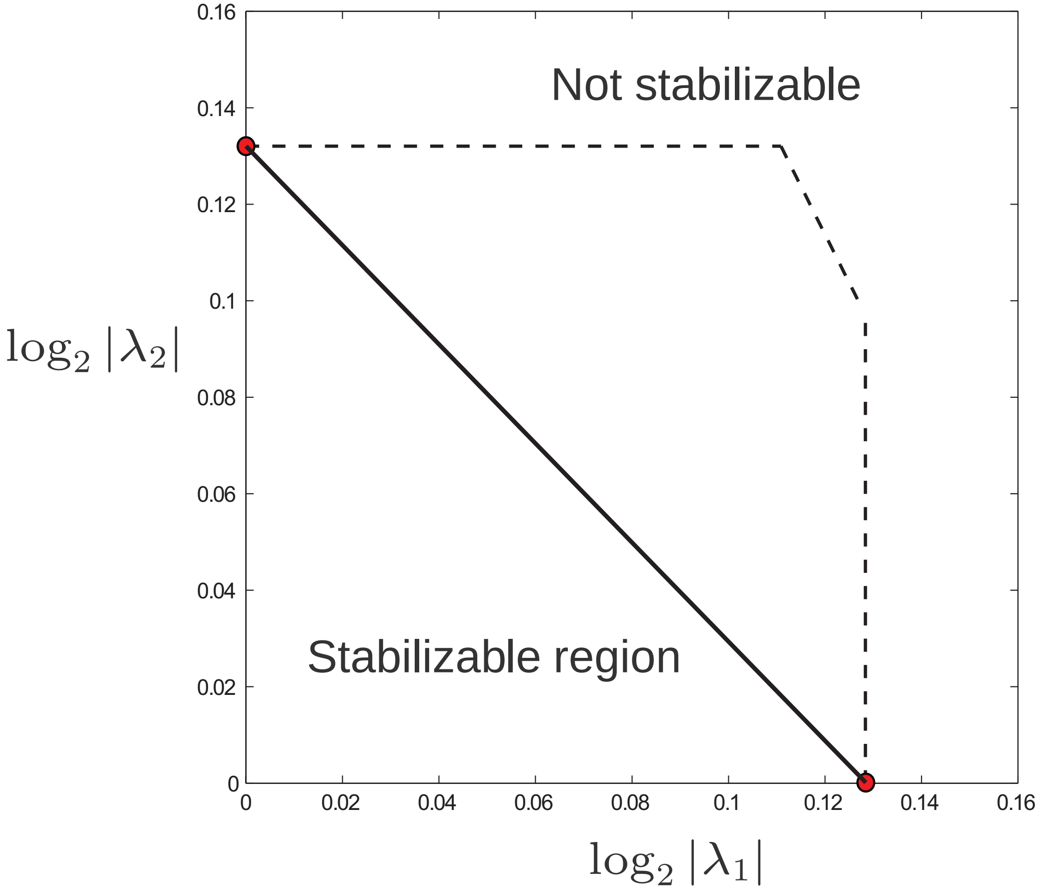
Vector case: sufficiency

- From the scalar case result we can achieve the two red points.



Vector case: sufficiency

- We can then time share between the two red points.



- Can we do better?

Sufficiency: rate allocation

Theorem 3. *Sufficient condition for stabilizability in the mean square sense is that* $(\log_2|\lambda_1|,\ldots,\log_2|\lambda_n|) \in \mathbb{R}^n_+$ *are inside the convex hull of the regions determined by the following* n *inequalities*

$$\mathbb{E}\left[\frac{|\lambda_i|^2}{2^{2\ \alpha_i(R)\ R}}\right] < 1, \quad i = 1,\ldots,n,$$

where the rate allocation vector $\boldsymbol{\alpha}(R) := [\alpha_1(R),\ldots,\alpha_n(R)]^T$ *satisfies*

$$\begin{cases} \alpha_i(r) \in [0,1] \\ \frac{r}{m_i}\,\alpha_i(r) \in \mathbb{N} \\ \sum_{i=1}^n \alpha_i(r) \le 1 \end{cases}$$

for all possible values r *that* R *can take (excluding 0).*

Sufficiency: rate allocation

Example 2. *Suppose the rate process is given by*

$$R = \begin{cases} 6 & \textit{w.p. } 1-p \\ 0 & \textit{w.p. } p, \end{cases}$$

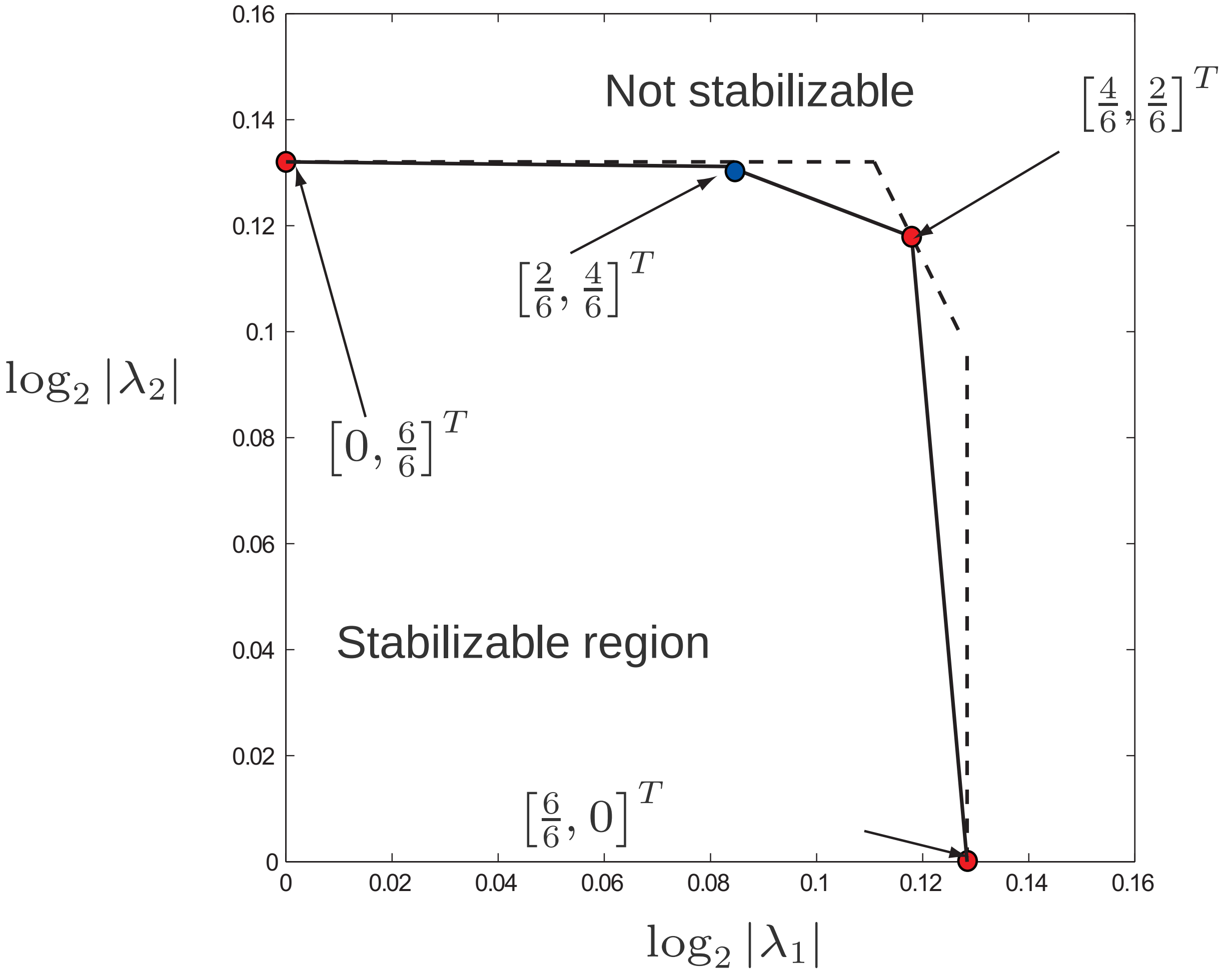
and, as in the previous example,

$$A = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}.$$

There are four (dominant) possible allocations $\boldsymbol{\alpha}(R) = [\alpha_1(R), \alpha_2(R)]^T$*:*

$$[\alpha_1(6), \alpha_2(6)]^T \in \left\{ \left[\frac{6}{6}, 0\right]^T, \left[\frac{4}{6}, \frac{2}{6}\right]^T, \left[\frac{2}{6}, \frac{4}{6}\right]^T, \left[0, \frac{6}{6}\right]^T \right\},$$

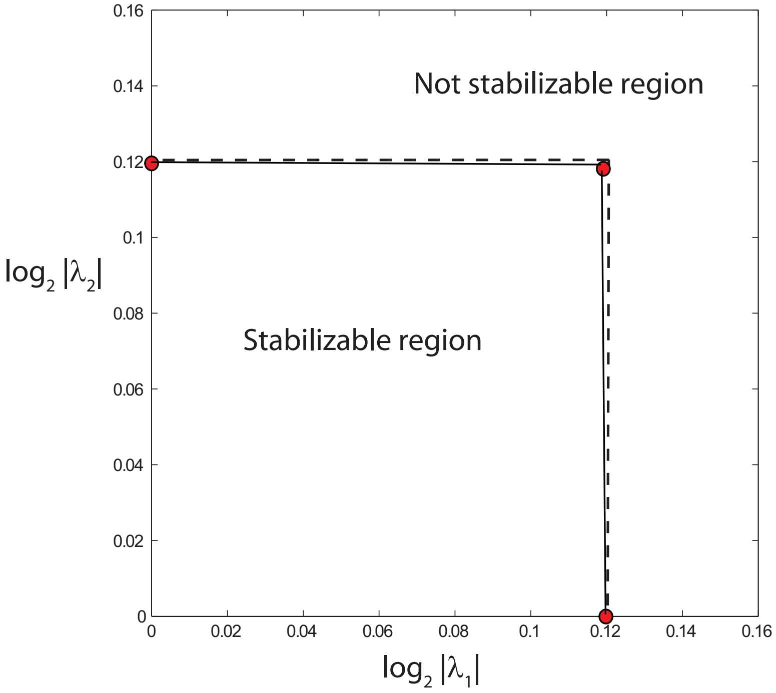
Sufficiency: example



- One point on the dominant face of the pentagon is optimal

Sufficiency: remarks

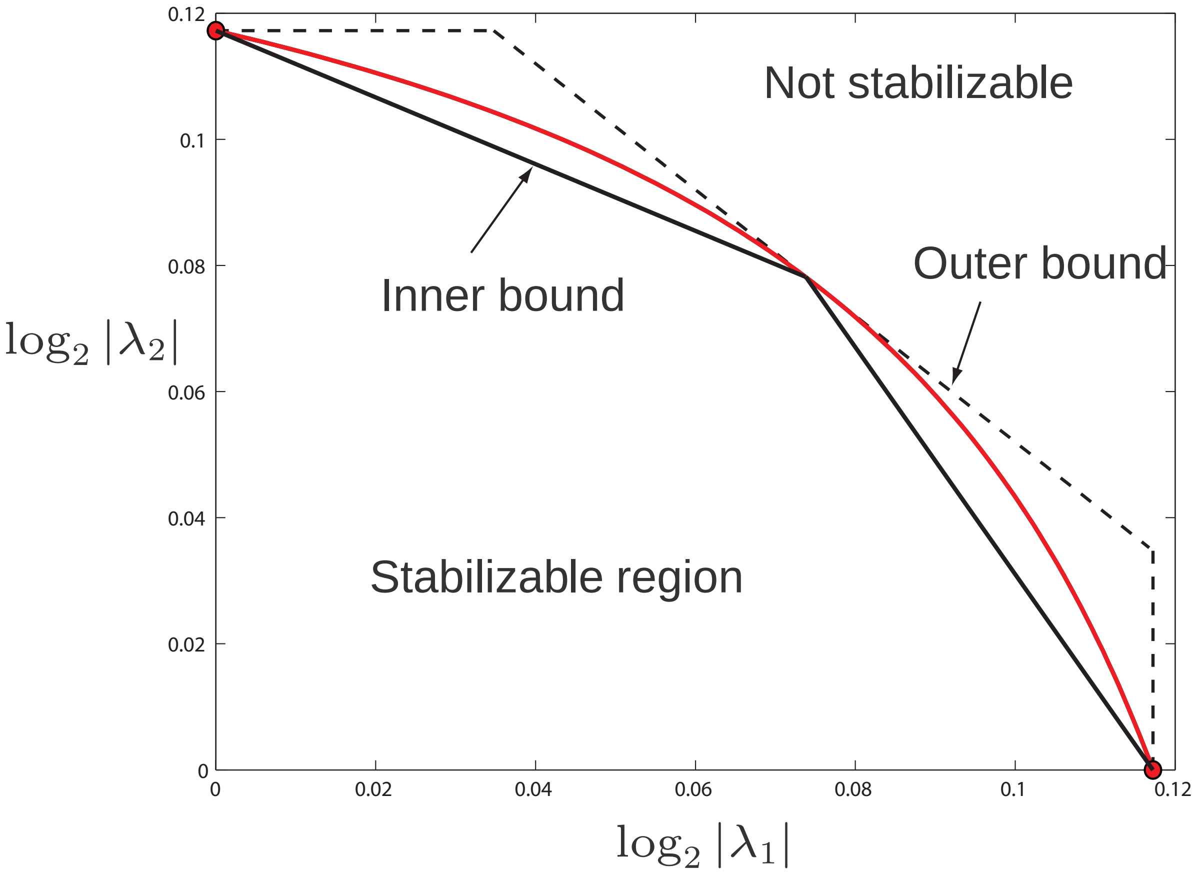
- One point on the dominant face of the pentagon is optimal
- If $R = \begin{cases} r & \text{w.p. } 1-p \\ 0 & \text{w.p. } p \end{cases}$, scheme asymptotically optimal as $r \to \infty$



- Scheme is optimal if pentagon reduces to hypercube or hyperplane

Remarks

- Above theorems provide polytopic inner and outer bounds to stabilizability region.



- Improved coding scheme for the case of an erasure channel.

Concluding Remarks

- Summary:
  - Unified approach to information-theoretic and packet loss models.
  - second moment stabilization with stochastic time-varying rates.
    - Scalar case: complete characterization.
    - Vector case: geometric structure, scheme is optimal in some limiting cases.
- Future directions:
  - Close the gap in the vector case
  - Model decoding errors